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Stochastic networks II

Problem set 4

Due date: May 22, 2012

Exercise 1

Let $\alpha > 0$. For every $k \geq 1$ put $r_k = \sqrt{\pi^{-1}\alpha\log(k)}$.

(a) show that for every $\alpha > 0$ there exists $\beta_1 > 0$ and $k_0 \geq 1$ such that for all $k \geq k_0$ the square $B_{1,k} = [-\sqrt{k}/2, \sqrt{k}/2]^2$ contains at least $\beta_1 k/\log(k)$ disjoint balls of radius r_k

(b) show that for every $\alpha > 0$ there exists $\beta_2 > 0$ such that for every $k > 0$ it is not possible to place at least $\beta_2 k/\log(k)$ disjoint balls of radius r_k inside $B_{1,k}$.

(c) show that there exists a sequence $\{\varepsilon_k\}_{k=2}^{\infty}$ in $(0,\infty)$ with the following properties

- $\varepsilon_k \to 0$ as $k \to \infty$
- $k/(\log k - \varepsilon_k) \in \mathbb{Z}$ for all $k \geq 2$.
- $B_{1,k}$ can be subdivided into $k/(\log k - \varepsilon_k)$ congruent squares of area $\log k - \varepsilon_k$

Exercise 2

Prove the existence of $\mu_0 \in (0,1)$ such that

$$\inf_{0<\mu<\mu_0} \liminf_{n\to\infty} \frac{\log(\mathbb{P}(\sum_{i=1}^n Y_i \geq n))}{n\log(\mu)} > 0,$$

where $\{Y_i\}_{i\geq 1}$ is a sequence of iid random variables with $Y_i \sim \mathbf{Poi}(\mu)$.

Hint. Use the explicit formula for the Laplace-transform of a Poisson-distributed random variable.

Exercise 3

Let $0 < r < r_c$ and let $X \subset \mathbb{R}^2$ be a homogeneous Poisson point process with intensity 1. For $n \geq 1$ write $p_n(r)$ for the probability that the connected component C_o of $G(X \cup \{o\}, r)$ containing the origin consists of precisely n vertices. Furthermore fix $\varepsilon > 0$ such that $r + 2\sqrt{2}\varepsilon < r_c$.

(a) define a site percolation process $Y = \{Y_z\}_{z\in\mathbb{Z}^2}$ on the graph $G(r+\sqrt{2}\varepsilon) = (\mathbb{Z}^2, E)$, where $E = \{\{z_1, z_2\} : |z_1 - z_2| \leq (r + \sqrt{2}\varepsilon)/\varepsilon\}$ by $Y_z = 1_{X(\varepsilon(z+[-1/2,1/2]^2))\geq 1}$. Denote by $C_o^{(1)}$ the activated site cluster at the origin. Prove

$$\limsup_{n\to\infty} \frac{\log\left(\mathbb{P}\left(\left|C_o^{(1)}\right| \geq n\right)\right)}{n} < 0$$

(b) prove the existence of $K > 0$ with

$$\limsup_{n\to\infty} \frac{\log\left(\mathbb{P}\left(\left|C_o^{(1)}\right| \leq n, |C_o| \geq Kn\right)\right)}{n} < 0$$

(c) conclude

$$\limsup_{n\to\infty} \frac{\log p_n(r)}{n} < 0$$

(d) show that there exists $r_0 \in (0,1)$ such that when choosing $\varepsilon = r$ we have

$$\inf_{0<r<r_0} \liminf_{n\to\infty} \frac{\log\left(\mathbb{P}\left(\left|C_o^{(1)}\right| \geq n\right)\right)}{\log(r)n} > 0$$

(e) conclude the existence of $r_0 \in (0,1)$ with

$$\inf_{0<r<r_0} \liminf_{n\to\infty} \frac{\log\left(\mathbb{P}\left(|C_o| \geq n\right)\right)}{\log(r)n} > 0$$

Hints. You may use the following results without proof.

- Let $0 < p < p_c$ and let $Y' = \{Y_z\}_{z\in\mathbb{Z}^2}$ be a Bernoulli site percolation process with activation probability p on the graph $G(r + \sqrt{2}\varepsilon) = (\mathbb{Z}^2, E)$, where $E = \{\{z_1, z_2\} : |z_1 - z_2| \leq (r + \sqrt{2}\varepsilon)/\varepsilon\}$. Denote by C_o' the activated site cluster at the origin. Then we have
$$\limsup_{n\to\infty} \frac{\log(\mathbb{P}(|C_o'| \geq n))}{n} < 0$$
- If $G = (V, E)$ is a countable connected graph with uniformly bounded vertex degree, then there exists $a > 1$ such that the number of connected subgraphs of G consisting of exactly n vertices and containing a fixed vertex $v_0 \in V$ is bounded from above by a^n.