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Stochastic networks II

Problem set 5

Due date: June 5, 2012

Exercise 1

Let $\lambda > 0$, $B \subset \mathbb{Z}^+$ and denote by $g_{\lambda,B} : \mathbb{Z}^+ \to [0, \infty)$ a function with the properties

- $g_{\lambda,B}(0) = 0$ and
- $\lambda g_{\lambda,B}(j + 1) - j g_{\lambda,B}(j) = 1_{j \in B} - \mathrm{Poi}_\lambda(B)$ for all $j \geq 0$

(a) Show that for $B, B' \subset \mathbb{Z}^+$ we have $g_{\lambda,B\cup B'} = g_{\lambda,B} + g_{\lambda,B'} - g_{\lambda,B\cap B'}$

(b) For $j \geq 0$ write $U_j = \{0, \ldots , j\}$. Prove

$$g_{\lambda,B}(j + 1) = \lambda^{-j-1} j! e^\lambda (\mathrm{Poi}_\lambda(B \cap U_j) - \mathrm{Poi}_\lambda(B)\mathrm{Poi}_\lambda(U_j)), \qquad \text{for all } j \geq 0.$$

(c) Let $B = \{0\}$. Show that $g_{\lambda,\{0\}}(k) > 0$ and $g_{\lambda,\{0\}}(k+1) - g_{\lambda,\{0\}}(k) < 0$ holds for all $k \geq 1$.

(d) Let $B = \{j\}$ with $j \geq 1$. Show that $g_{\lambda,\{j\}}(k + 1) - g_{\lambda,\{j\}}(k) < 0$ holds for all $k \in \{0, \ldots , j - 1\}$.

(e) Let $B = \{j\}$ with $j \geq 1$. Show that $g_{\lambda,\{j\}}(j + 1) - g_{\lambda,\{j\}}(j) > 0$.

(f) Let $B = \{j\}$ with $j \geq 1$. Show that $g_{\lambda,\{j\}}(k + 1) - g_{\lambda,\{j\}}(k) < 0$ holds for all $k \geq j + 1$.

Exercise 2

Let $d, n \geq 1$ and let $Y_1, \ldots , Y_n$ be iid with $Y_i \sim U(\{1, \ldots , d\})$. Write $I = \{\alpha \subset \{1, \ldots , n\} : |\alpha| = 2\}$ and for $\alpha = \{i, j\} \in I$ write X_α for the indicator function of the event $Y_i = Y_j$. Finally put $X = \sum_{\alpha \in I} X_\alpha$.

(a) For $\alpha \in I$ write $I_\alpha = \{\beta \in I : \alpha \cap \beta \neq \emptyset\}$. Prove that X_α is independent of the random vector $(X_\gamma)_{\gamma \notin I_\alpha}$.

(b) Show

$$\sum_{\alpha\in I} \sum_{\beta\in I_\alpha} \mathbb{E}X_\alpha \mathbb{E}X_\beta = \binom{n}{2} \cdot \left(\binom{n}{2} - \binom{n-2}{2}\right) d^{-2}.$$

(c) Show that for all $\alpha \neq \beta$ we have $\mathbb{E}(X_\alpha X_\beta) = \mathbb{E}X_\alpha \mathbb{E}X_\beta$.

(d) Conclude

$$d_{TV}(P_X, \mathrm{Poi}_\lambda) \leq \lambda^{-1}(1 - e^{-\lambda})d^{-2}\binom{n}{2}(4n - 7),$$

where $\lambda = \binom{n}{2} d^{-1}$.

Hint. Use corollary 3.4 in the lecture notes.

Exercise 3

Let $p \in (0,1)$, $n \geq 1$ and $1 \leq k \leq n$. Let $Y_1, \dots, Y_n$ be iid Bernoulli random variables with $\mathbb{P}(Y_1 = 1) = p$. In this problem all indices are considered modulo n. Furthermore with $I = \{1, \dots, n\}$ and $i \in I$ write $X_i = \prod_{j=i}^{i+k-1} Y_j$ and put $X = \sum_{i=1}^{n} X_i$. That is, X counts the number of so-called *runs* of length $\geq k$.

(a) For $i \in I$ write $I_i = \{i-(k-1), \dots, i+k-1\}$. Prove that X_i is independent of the random vector $(X_j)_{j \notin I_i}$

(b) Prove $\sum_{i \in I} \sum_{j \in I_i \setminus \{i\}} \mathbb{E}(X_i X_j) = 2np^k \sum_{i=1}^{k-1} p^i$

(c) Conclude

$$d_{TV}(P_X, \mathrm{Poi}_\lambda) \leq \lambda^{-1}(1 - e^{-\lambda}) \left((2k-1)\lambda p^k + 2\lambda \sum_{i=1}^{k-1} p^i \right),$$

where $\lambda = np^k$ and deduce $\sup_{n \geq k} d_{TV}(P_X, \mathrm{Poi}_\lambda) \to 0$ as $p \to 0$.

Hint. Use corollary 3.4 in the lecture notes.

Exercise 4

Consider an $n \times n$ grid on the torus. In particular, this grid has n^2 vertices each having degree 4. Consider a Bernoulli bond process with parameter p on the set of the $2n^2$ edges of this grid. Write $I = \{1, \dots, n^2\}$ and for $i \in I$ denote by X_i the indicator variable that none of the 4 adjacent edges to the node i is activated. Furthermore write $X = \sum_{i=1}^{n^2} X_i$ for the number of isolated nodes.

(a) For $i \in I$ write I_i for the union of the set of neighbors of the i-th node and $\{i\}$. Prove that X_i is independent to the random vector $(X_j)_{j \notin I_i}$

(b) Prove $\sum_{i \in I} \sum_{j \in I_i \setminus \{i\}} \mathbb{E}(X_i X_j) = 4n^2(1-p)^7$

(c) Conclude

$$d_{TV}(P_X, \mathrm{Poi}_\lambda) \leq 5(1-p)^4 + 4(1-p)^3.$$

Hint. Use corollary 3.4 in the lecture notes.