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SS 2012
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Stochastic networks II

Problem set 6

Due date: June 12, 2012

Exercise 1

Let $r > 0$ and let $X \subset \mathbb{R}^2$ be a homogeneous Poisson point process with intensity 1. Write $N \in \mathbb{Z}^+ \cup \{\infty\}$ for the (random) number of infinite connected components of the graph $G(X, r)$.

(a) show that N is a.s. constant, i.e. there exists $n \in \mathbb{Z}^+ \cup \{\infty\}$ such that $\mathbb{P}(N = n) = 1$.

(b) show that if $\mathbb{P}(N < \infty) = 1$, then there exists $k > 0$ such that the probability that the square $[-k, k]^2$ intersects all infinite connected components is positive.

(c) show that if $\mathbb{P}(N < \infty) = 1$ and $r > r_c$, then $\mathbb{P}(N = 1) > 0$.

(d) conclude that N is a.s. equal to one of the values 0, 1 or ∞.

Hint. Use the ergodicity of X for part (a).

Exercise 2

Let X be a homogeneous Poisson point process with intensity 1 and write $(\mathbb{N}, \mathcal{N}, \mathbb{P})$ for its canonical probability space. Furthermore $f : \mathbb{N} \to [0, \infty)$ is said to be increasing if for any $\varphi, \varphi' \in \mathbb{N}$ with $\varphi \subset \varphi'$ we have $f(\varphi) \le f(\varphi')$. Denote by $\mathcal{F}_n$ the σ-algebra generated by the random variables $\left\{1_{X\left(3^{-n}z+]-3^{-n}/2,3^{-n}/2]^2\right)\ge 1}\right\}_{z\in\mathbb{Z}^2\cap[-3^n n,3^n n]^2}$.

(a) show that the $(\mathcal{F}_n)_{n\ge1}$ form a filtration with $\sigma\left(\bigcup_{n=1}^{\infty}\mathcal{F}_n\right) = \mathcal{N}$.

(b) if $f : \mathbb{N} \to [0, \infty)$ is an increasing function, show that $\mathbb{E}(f \mid \mathcal{F}_k)$ defines an increasing function on the discrete product space $\{0, 1\}^{\{\mathbb{Z}^2\cap[-3^n n,3^n n]^2\}}$.

(c) prove that if $f_1, f_2 : \mathbb{N} \to [0, \infty)$ are increasing, then $\mathbb{E}\left(\mathbb{E}\left(f_1 \mid \mathcal{F}_k\right)\mathbb{E}\left(f_2 \mid \mathcal{F}_k\right)\right) \ge \mathbb{E}\left(f_1\right)\mathbb{E}\left(f_2\right)$.

(d) using the martingale convergence theorem conclude that $\mathbb{E}\left(f_1 f_2\right) \ge \mathbb{E}\left(f_1\right)\mathbb{E}\left(f_2\right)$.

Hint. Use Lemma 2.5 from the lecture notes for part (c).

Exercise 3

Let X be a homogeneous Poisson point process with intensity 1 and let $r > r_c$.

(a) show that there exists $c > 0$ such that for all $n \ge 1$ the probability $G(X, r)$ contains a closed curve inside $[-3^{n+1}/2, 3^{n+1}/2]^2$ surrounding $[-3^n/2, 3^n/2]^2$ is bounded from below by c.

(b) show that the event of part (a) occurs a.s. for infinitely many values of n.

(c) conclude that $G(X, r)$ contains a.s. exactly 1 infinite cluster.

Hint. Use Theorem 3.15 from the lecture notes and Exercise 2 for part (a).