Prof. Dr. V. Schmidt
C. Hirsch

SS 2012
June 12, 2012

# Stochastic networks II

## Problem set 7

Due date: June 19, 2012

**Exercise 1**

Let $r > 0$, let $k \geq 1$ and let $X \subset \mathbb{R}^2$ be a homogeneous Poisson point process with intensity 1. For $s > 0$ write $B_s = [-\sqrt{s}/2, \sqrt{s}/2]^2$ and denote by $M_s$ the number of points $X_i \in X \cap B_s$ such that the connected component of $G(X, r)$ containing $X_i$ consists of exactly $k$ vertices. Similarly write $\widetilde{M_s}$ for the number of points $X_i \in X \cap B_s$ such that the connected component of $G(X, r)$ containing $X_i$ consists of exactly $k$ vertices and such that additionally $X_i$ is the vertex with smallest $x$-coordinate in this component. Finally denote by $p_k(r)$ resp. $\widetilde{p_k}(r)$ the probabilities that the connected component $C_o$ of $G(X \cup \{o\}, r)$ containing the origin $o$ consists of exactly $k$ vertices, respectively that $|C_o| = k$ and additionally the origin $o$ is the vertex with smallest $x$-coordinate in this component.

(a) Prove that $\left| M_s - k\widetilde{M_s} \right| \leq X\left(B_{(\sqrt{s}+kr)^2} \setminus B_{(\sqrt{s}-kr)^2}\right)$ and conclude $s^{-1}\mathbb{E}\left| M_s - k\widetilde{M_s} \right| \to 0$ as $s \to \infty$.

(b) Show that $\mathbb{E}M_s = sp_k(r)$ and $\mathbb{E}(\widetilde{M_s}) = s\widetilde{p_k}(r)$.

(c) Conclude $p_k(\lambda) = k\widetilde{p_k}(\lambda)$.

**Exercise 2**

Let $X \subset \mathbb{R}^2$, $k \geq 1$, $r > 0$, $\widetilde{p_k}(r)$ and $p_k(r)$ be as in Problem 1. Furthermore write $B = B_{r(k+3)}(o)$ for the ball of radius $r(k+3)$ centered at the origin and for $\psi \subset \varphi \subset \mathbb{R}^2$ finite write $g(\psi, \varphi)$ for the indicator function of the event that $G(\psi \cup \{o\}, r)$ forms a connected component of $G(\varphi \cup \{o\}, r)$ consisting of exactly $k+1$ vertices and such that the origin is the vertex with the smallest $x$-coordinate in this component.

(a) Let $Y = \{Y_1, \ldots, Y_k\}$ be a set of iid random vectors distributed uniformly in $B$. Prove

$$\widetilde{p_{k+1}}(r) = \frac{\left(\pi(k+3)^2 r^2\right)^k}{k!}\mathbb{E}g(Y, Y \cup (X \cap B)).$$

(b) Write $\widetilde{h}(x_1, \ldots, x_k)$ for the indicator function of the event that $G(\{o, x_1, \ldots, x_k\}, 1)$ is connected and that the origin is the vertex with the smallest $x$-coordinate. Furthermore write

$$A(o, x_1, \ldots, x_k) = \nu_2\left(B_r(o) \cup \bigcup_{i=1}^{k} B_r(x_i)\right).$$

Using this notation, prove that

$$\widetilde{p_{k+1}}(r) = \frac{1}{k!}\int_B \cdots \int_B \widetilde{h}(x_1, \ldots, x_k)\exp(-A(o, x_1, \ldots, x_k))dx_1 \cdots dx_k.$$

(c) Finally write $h(x_1, \ldots, x_k) = \widetilde{h}(x_1, \ldots, x_k) \cdot 1_{\pi_1(o)<\pi_1(x_1)<\ldots<\pi_1(x_k)}$, where $\pi_1(x_i)$ denotes the first coordinate of $x_i$. Prove that

$$p_{k+1}(r) = (k+1)\int_B \cdots \int_B h(o, x_1, \ldots, x_k)\exp(-A(o, x_1, \ldots, x_k))dx_1 \cdots dx_k.$$

**Exercise 3**

Let $\varphi \subset \mathbb{R}^d$ be finite and with the property that for all $x_1, x_2, y_1, y_2 \in \varphi$ with $x_1 \neq x_2$, $y_1 \neq y_2$ and $\{x_1, x_2\} \neq \{y_1, y_2\}$ we have $|x_1 - x_2| \neq |y_1 - y_2|$. Define a graph $G_1$ on the vertex set $\varphi$ by drawing an edge between $x$ and $y$ if and only if there do not exist $x = x_0, x_1, \ldots, x_n = y$ such that $|x - y| > |x_i - x_{i+1}|$ holds for all $0 \leq i \leq n-1$. Furthermore define a graph $G_2$ on the vertex $W$ by drawing an edge between $x$ and $y$ if and only if there exist $W \subset \varphi$ such that $|\{x, y\} \cap W| = 1$ and $|x - y| \leq |x' - y'|$ for all $x' \in W$ and $y' \in \varphi \setminus W$.

(a) Prove $G_1 = MST(\varphi)$.

(b) Prove $G_2 = MST(\varphi)$.