Stochastic networks II
Problem set 7
Due date: June 19, 2012

Exercise 1

Let $r > 0$, let $k \geq 1$ and let $X \subset \mathbb{R}^2$ be a homogeneous Poisson point process with intensity 1. For $s > 0$ write $B_s = [-\sqrt{s}/2, \sqrt{s}/2]^2$ and denote by $M_s$ the number of points $X_i \in X \cap B_s$ such that the connected component of $G(X, r)$ containing $X_i$ consists of exactly $k$ vertices. Similarly write $\tilde{M}_s$ for the number of points $X_i \in X \cap B_s$ such that the connected component of $G(X, r)$ containing $X_i$ consists of exactly $k$ vertices and such that additionally $X_i$ is the vertex with smallest $x$-coordinate in this component. Finally denote by $p_k(r)$ resp. $\tilde{p}_k(r)$ the probabilities that the connected component $C_o$ of $G(X \cup \{o\}, r)$ containing the origin $o$ consists of exactly $k$ vertices, respectively that $|C_o| = k$ and additionally the origin $o$ is the vertex with smallest $x$-coordinate in this component.

(a) Prove that $|M_s - k\tilde{M}_s| \leq X(B(\sqrt{s}+kr)^2 \setminus B(\sqrt{s}-kr)^2)$ and conclude $s^{-1}E|M_s - k\tilde{M}_s| \to 0$ as $s \to \infty$.

(b) Show that $E(M_s) = sp_k(r)$ and $E(\tilde{M}_s) = s\tilde{p}_k(r)$.

(c) Conclude $p_k(\lambda) = k\tilde{p}_k(\lambda)$.

Exercise 2

Let $X \subset \mathbb{R}^2$, $k \geq 1$, $r > 0$, $\tilde{p}_k(r)$ and $p_k(r)$ be as in Problem 1. Furthermore write $B = B_{r(k+3)}(o)$ for the ball of radius $r(k+3)$ centered at the origin and for $\psi \subset \varphi \subset \mathbb{R}^2$ finite write $g(\psi, \varphi)$ for the indicator function of the event that $G(\psi \cup \{o\}, r)$ forms a connected component of $G(\varphi \cup \{o\}, r)$ consisting of exactly $k+1$ vertices and such that the origin is the vertex with the smallest $x$-coordinate in this component.

(a) Let $Y = \{Y_1, \ldots, Y_k\}$ be a set of iid random vectors distributed uniformly in $B$. Prove

$$\tilde{p}_{k+1}(r) = \frac{(\pi(k+3)^2r^2)^k}{k!}Eg(Y, Y \cup (X \cap B)).$$

(b) Write $\tilde{h}(x_1, \ldots, x_k)$ for the indicator function of the event that $G(\{o, x_1, \ldots, x_k\}, 1)$ is connected and that the origin is the vertex with the smallest $x$-coordinate. Furthermore write

$$A(o, x_1, \ldots, x_k) = \nu_2\left(B_r(o) \cup \bigcup_{i=1}^k B_r(x_i)\right).$$
Using this notation, prove that

\[ \tilde{p}_{k+1}(r) = \frac{1}{k!} \int_{B} \ldots \int_{B} \tilde{h}(x_1, \ldots, x_k) \exp(-A(o, x_1, \ldots, x_k)) \, dx_1 \ldots dx_k. \]

(c) Finally write \( h(x_1, \ldots, x_k) = \tilde{h}(x_1, \ldots, x_k) \cdot 1_{\pi_1(o) < \pi_1(x_1) < \ldots < \pi_1(x_k)} \), where \( \pi_1(x_i) \) denotes the first coordinate of \( x_i \). Prove that

\[ p_{k+1}(r) = (k + 1) \int_{B} \ldots \int_{B} h(o, x_1, \ldots, x_k) \exp(-A(o, x_1, \ldots, x_k)) \, dx_1 \ldots dx_k. \]

Exercise 3

Let \( \varphi \subset \mathbb{R}^d \) be finite and with the property that for all \( x_1, x_2, y_1, y_2 \in \varphi \) with \( x_1 \neq x_2, y_1 \neq y_2 \) and \( \{x_1, x_2\} \neq \{y_1, y_2\} \) we have \( |x_1 - x_2| \neq |y_1 - y_2| \). Define a graph \( G_1 \) on the vertex set \( \varphi \) by drawing an edge between \( x \) and \( y \) if and only if there do not exist \( x = x_0, x_1, \ldots, x_n = y \) such that \( |x - y| > |x_i - x_{i+1}| \) holds for all \( 0 \leq i \leq n - 1 \). Furthermore define a graph \( G_2 \) on the vertex \( W \) by drawing an edge between \( x \) and \( y \) if and only if there exist \( W \subset \varphi \) such that \( |\{x, y\} \cap W| = 1 \) and \( |x - y| \leq |x' - y'| \) for all \( x' \in W \) and \( y' \in \varphi \setminus W \).

(a) Prove \( G_1 = MST(\varphi) \).

(b) Prove \( G_2 = MST(\varphi) \).