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# Solution sketches for selected problem sets

## Problem set 7, Exc. 2

*a.* Observe we have $g(\psi, \varphi) = g(\psi, B \cap \varphi)$ and that $g(\psi, \varphi)$ can only be non-zero if $\psi \subset B$. In particular, by applying the Slivnyak-Mecke formula we obtain

$$\begin{aligned}
\widetilde{p_{k+1}}(r) &= \frac{1}{k!}\mathbb{E} \sum_{X_1,\ldots,X_k \in X} g(\{X_1,\ldots,X_k\}, X \cap B) \\
&= \frac{1}{k!}\int_{\mathbb{R}^2} dy_1 \cdots \int_{\mathbb{R}^2} dy_k \mathbb{E}\left(g(\{y_1,\ldots,y_k\}, (X \cap B) \cup \{y_1,\ldots,y_k\})\right) \\
&= \frac{1}{k!}\int_{B} dy_1 \cdots \int_{B} dy_k \mathbb{E}\left(g(\{y_1,\ldots,y_k\}, (X \cap B) \cup \{y_1,\ldots,y_k\})\right) \\
&= \frac{\nu_2(B)^k}{k!}\int_{B} \frac{1}{\nu_2(B)} dy_1 \cdots \int_{B} dy_k \frac{1}{\nu_2(B)} \mathbb{E}\left(g(\{y_1,\ldots,y_k\}, (X \cap B) \cup \{y_1,\ldots,y_k\})\right) \\
&= \frac{(\pi(k+3)^2 r^2)^k}{k!}\mathbb{E}\left(g(Y, (X \cap B) \cup Y)\right)
\end{aligned}$$

*b.* As before we use the Slivnyak-Mecke formula to compute.

$$\begin{aligned}
\widetilde{p_{k+1}}(r) &= \frac{1}{k!}\int_{B} dy_1 \cdots \int_{B} dy_k \mathbb{E}\left(g(\{y_1,\ldots,y_k\}, (X \cap B) \cup \{y_1,\ldots,y_k\})\right) \\
&= \frac{1}{k!}\int_{B} dy_1 \cdots \int_{B} dy_k \widetilde{h}(y_1,\ldots,y_k) \exp\left(-\nu_2(B_r(o) \cup \bigcup_{i=1}^{k} B_r(y_i))\right) \\
&= \frac{1}{k!}\int_{B} dx_1 \cdots \int_{B} dx_k \widetilde{h}(x_1,\ldots,x_k) \exp\left(-A(o, x_1,\ldots,x_k)\right)
\end{aligned}$$

*c.* Observe that applying any permutation of the values of $(x_1, \ldots, x_k)$ in the integral formula in part $(b)$ does not change the value of the integral. In particular we may also only integrate over the set $\{\pi_1(o) < \pi_1(x_1) < \ldots < \pi_1(x_k)\}$ and multiply the result by $k!$ afterwards. In particular, we have

$$\widetilde{p_{k+1}}(r) = \int_B dx_1 \cdots \int_B dx_k h(x_1,\ldots,x_k) \exp\left(-A(o, x_1,\ldots,x_k)\right).$$

Combining this result with the relation $p_{k+1}(r) = (k+1)\widetilde{p_{k+1}}(r)$

# Problem set 8, Exc. 2b

We compute

$$\begin{aligned}\mathbb{P}(A_n^c) &\le \sum_{i=1}^{25} \mathbb{P}(X(K_i) = 0)\\ &= 25\mathbb{P}(X(Q_n) = 0)\\ &= 25\exp(-n^2),\end{aligned}$$

so that

$$\begin{aligned}\sum_{n\ge1} \mathbb{P}(A_n^c) &\le 25 \sum_{n\ge1} e^{-n}\\ &= \frac{25e^{-1}}{1-e^{-1}}\\ &< \infty\end{aligned}$$

In particular, with probability 1 there exists $N \ge 1$ such that for all $n \ge N$ the centerr of every Voronoi cell intersecting $Q_n(o)$ is contained in $Q_{5n}(o)$. Since any bounded set $B \subset \mathbb{R}^2$ is contained in $Q_{n_1}(o)$ for some $n_1$ sufficiently large we see that the number of cells intersecting $B$ is bounded from above by $X(Q_{5n_1}(o)) < \infty$. Since any edge is adjacent to two Voronoi cells, we see that the number of edges intersecting $B$ is bounded from above by $X(Q_{5n_1}(o))^2$.

# Problem set 8, Exc. 3

*a.* First observe that $E_\cup$ is closed. Indeed, let $x \in \mathbb{R}^2 \setminus E_\cup$. Then, by local finiteness, $B_1(x)$ intersects only finitely many edges $e_1, \dots, e_m$ of $G$. Denote by $r$ half the minimal distance of $x$ to one of $e_1, \dots, e_m$. Then $B_r(x) \cap E_\cup = \emptyset$, so that $\mathbb{R}^2 \setminus E_\cup$ is open.

Now let $C \subset \mathbb{R}^2$ be a connected compoonent of $\mathbb{R}^2 \setminus E_\cup$. In particular, $C$ is both open and closed in the trace topology of $\mathbb{R}^2 \setminus E_\cup$. Since $\mathbb{R}^2 \setminus E_\cup$ is open, we conclude also that $C$ is open.

Finally let $y \in \partial C$ be arbitrary and suppose $y \notin E_\cup$. Then choose $r > 0$ such that $B_r(y) \cap E_\cup = \emptyset$. In particular, all elements of $B_r(y)$ are contained in the same connected component of $\mathbb{R}^2 \setminus E_\cup$. However, by assumption we have $B_r(y) \cap C \ne \emptyset$ and $B_r(y) \cap (\mathbb{R}^2 \setminus C) \ne \emptyset$ which yields a contradiction to the assumption that $C$ is a connected component.

*b.*Let $C$ be a cell as in the definition given in the lecture notes. By assumption $C$ is connected. Denote by $C' \subset \mathbb{R}^2 \setminus E_\cup$ the connected component of $\mathbb{R}^2 \setminus E_\cup$ containing $C$. Assume $x_0 \in C' \setminus C$ and choose $x_1 \in C$ and a path $\gamma \subset C'$ connecting $x_0$ and $x_1$. Let $x_2$ be the last point of $\gamma$ such that all previous points lie in $C$. In particular, we conclude $x_2 \in \partial C$, i.e. $P \in E_\cup$. However, this is a contradiction to the assumption that $\gamma$ lies in $\mathbb{R}^2 \setminus E_\cup$.

# Problem set 9, Exc. 2

*a.*Using the Slivnyak-Mecke formula and stationarity of $X$ we compute

$$\begin{aligned} a^{-2}\mathbb{E}\sum_{S_n\in X\cap[-a/2,a/2]^2} deg_{Del(X)}(S_n) &= a^{-2}\int dx 1_{x\in[-a/2,a/2]^2}\mathbb{E}deg_{Del(X\cup\{x\})}(x) \\ &= a^{-2}\int dx 1_{x\in[-a/2,a/2]^2}\mathbb{E}deg_{Del(X\cup\{o\})}(o) \\ &= a^{-2}\nu_2([-a/2,a/2]^2)\mathbb{E}deg_{Del(X\cup\{o\})}(o) \\ &= \mathbb{E}deg_{Del(X\cup\{o\})}(o) \end{aligned}$$

*b.* Consider the graph $G$ formed by triangles intersecting $[-n/2,n/2]^2$. Then we use a number of notations

(a) the number of Delaunay triangles in $G$ is denoted by $f$

(b) the number of vertices in $G$ is denoted by $m$

(c) the number of edges in $G$ is denoted by $e$

(d) the number of edges in $G$ such that both endpoints lie in $Q_n$ is denoted by $e_1$

(e) the number of edges in $G$ such that exactly one endpoint lies in $Q_n$ is denoted by $e_2$

(f) the number of edges in $G$ such that none of its endpoint lies in $Q_n$ is denoted by $e_3$

Note that since $G$ is a planar graph we may apply Euler's formula to obtain

$$e = f + m - 1, \tag{1}$$

(note that $f$ does not contain the "outer" face which is unbounded). Furthermore it is easy to check that we have

$$\sum_{S_i\in X\cap Q_n} deg_{Del(X)}(S_i) = 2e_1 + e_2, \tag{2}$$

and

$$3f = 2e_1 + 2e_2 + e_3. \tag{3}$$

From (1) and (3) we obtain

$$e = \frac{2}{3}e - \frac{1}{3}e_3 + m - 1 \tag{4}$$

or equivalently

$$e = -e_3 + 3m - 3. \tag{5}$$

Finally combining (5) and (2) we obtain

$$\sum_{S_i\in X\cap Q_n} deg_{Del(X)}(S_i) = -e_2 - 4e_3 + 6m - 6. \tag{6}$$

In particular, we obtain

$$\left|\frac{1}{6}\sum_{S_i\in X\cap Q_n} deg_{Del(X)}(S_i) - X([-n/2,n/2]^2)\right| = \left|m - X([-n/2,n/2]^2) - 1 - \frac{1}{6}e_2 - \frac{2}{3}e_3\right| \leq 5Y_n.$$

For the last inequality we made use of the following easy relations.

(a) $e_2 \leq 2Y_n$

(b) $e_3 \leq Y_n$

(c) $|m - X([-n/2,n/2]^2)| \leq Y_n$

*c.* Using part (*a*) and (*b*) we compute

$$\begin{aligned}
&\left|\frac{1}{6}\mathbb{E}deg_{Del(X\cup\{o\})}(o) - 1\right| \\
&\leq \left|\frac{n^{-2}}{6}\mathbb{E}\sum_{S_i\in X\cap Q_n} deg_{Del(X)}(S_i) - n^{-2}\mathbb{E}X([-n/2,n/2]^2)\right| \\
&+ \left|n^{-2}\mathbb{E}X([-n/2,n/2]^2) - 1\right| \\
&\leq \frac{5n^{-2}}{6}\mathbb{E}Y_n \\
&+ \left|n^{-2}\mathbb{E}X([-n/2,n/2]^2) - 1\right|
\end{aligned}$$

By ergodicity the second part tends to 0 so that it suffices to show $n^{-2}\mathbb{E}Y_n \to 0$. If we denote (as in part (*b*)) by $e_2 = e_2(n)$ the number of edges in $Del(X)$ such that exactly one end-point is contained in $[-n/2,n/2]^2$, then it is easy to see that we have $Y_n \leq 2e_2(n)$. Furthermore we denote by $Z_n^{(1)}$ the sum of degrees of points in $X \cap (Q_n(o) \setminus Q_{n-\sqrt{n}}(o))$ and by $Z_n^{(2)}$ the number of edges such that one end point is contained in $Q_{n-\sqrt{n}}(o)$ and the other one in $\mathbb{R}^2 \setminus Q_n(o)$, then it is easy to check that we have $e_2(n) \leq Z_n^{(1)} + Z_n^{(2)}$. We consider the two summands separately.

First we compute

$$\begin{aligned}
n^{-2}\mathbb{E}Z_n^{(1)} &= n^{-2}\mathbb{E}\sum_{X_i\in X\cap(Q_n(o)\setminus Q_{n-\sqrt{n}}(o))} deg_{Del(X)}(X_i) \\
&= n^{-2}\nu_2(Q_n(o)\setminus Q_{n-\sqrt{n}}(o))\mathbb{E}deg_{Del(X)}(o)
\end{aligned}$$

and the latter expression tends to 0 as $n \to \infty$.

For $Z_n^{(2)}$ we compute

$$\begin{aligned}
\mathbb{E}Z_n^{(2)} &= \mathbb{E} \sum_{X_i \in X \cap Q_n(o)} \sum_{X_j \in X} 1_{\{X_i, X_j\} \in Del(X)} 1_{|X_i - X_j| \geq \sqrt{n}} \\
&= \int dx \int dy 1_{x \in Q_n(o)} 1_{|x-y| \geq \sqrt{n}} \mathbb{E}\left(1_{\{x,y\} \in Del(X \cup \{x,y\})}\right) \\
&\leq \int dx \int dy 1_{x \in Q_n(o)} 1_{|x-y| \geq \sqrt{n}} \exp(-\pi \left|x - y\right|^2 /4) \\
&= \int dx \int dz 1_{x \in Q_n(o)} 1_{|z| \geq \sqrt{n}} \exp(-\pi \left|z\right|^2 /4) \\
&= 2\pi \int dx \int_{\sqrt{n}}^{\infty} dr 1_{x \in Q_n(o)} r \exp(-\pi r^2/4) \\
&= 2\pi n^2 \int_{\sqrt{n}}^{\infty} dr r \exp(-\pi r^2/4) \\
&= 4\pi n^2 \exp(-\pi n/4),
\end{aligned}$$

and the latter expresion tends to 0 as $n \to \infty$.