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Dr. Tim Brereton
Matthias Neumann

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Methods of Monte Carlo Simulation II
Exercise Sheet 4

Deadline: Mai 22, 2014 at 1pm before the exercises

Please hand in a printed version of your Matlab code and the output of the programs

The solution of the additional exercise for the reading course will be presented on Mai 22, 2014 at 12.30 pm

Exercise 1 (1+1+3+2)

Let $\{X_n\}_{n\geq 0}$ be a Markov chain with transition matrix

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & p & 1-p \\ \frac{p}{2} & 1-\frac{p}{2} & 0 \end{pmatrix},$$

where $p \in [0,1]$ and arbitrary initial distribution λ.

a) For which p is $\{X_n\}_{n\geq 0}$ irreducible?

b) For which p is $\{X_n\}_{n\geq 0}$ recurrent?

c) For which p does $\{X_n\}_{n\geq 0}$ have a stationary distribution? Compute the stationary distribution π if it exists.

d) Let $p = 0.5$. Write a Matlab program for estimating the expectation of the first passage times $\mathbb{E}_i \widetilde{\tau}_i$ for $i \in \{1,2,3\}$ based on 10000 realizations of the Markov chain.

Exercise 2 (3)

Let $\{X_n\}_{n\geq 0}$ be a stochastic process such that $X_0, X_1 - X_0, X_2 - X_1, \ldots$ are i.i.d. discrete random variables. Show that $\{X_n\}_{n\geq 0}$ is a Markov chain.

Exercise 3 (2+3)

Let $\{S_n\}_{n\geq 1}$ be a sequence of i.i.d. random variables with $S_1 \sim \text{Exp}(\lambda)$ with $\lambda > 0$. Define the process $\{X_n\}_{n\geq 0}$ by $X_0 = 0$ and

$$X_n = \sum_{i=1}^{\infty} \mathbb{I}\{S_1 + \ldots + S_i \leq n\},$$

for each $n \geq 1$.

a) Show that $\mathbb{P}(S_1 > s + t \mid S_1 > s) = \mathbb{P}(S_1 > t)$ for all $s, t \geq 0$.

b) Show that $\{X_n\}_{n\geq 0}$ is a Markov chain.

Exercise 4 (Additional exercise for the reading course)

Let $\{X_n\}_{n\geq 1}$ be a sequence of i.i.d. random variables with $X_1 \sim U([0,1])$. We want to estimate

$$p = \mathbb{P}\left(X_1 + \ldots + X_n \geq n\left(2 - \frac{1}{\log(2)}\right)\right)$$

by means of important sampling.

a) Compute

$$g_\theta(x) = \frac{e^{\theta x}\mathbb{I}\{0 \leq x \leq 1\}}{\mathbb{E}e^{\theta X_1}}$$

b) Choose $g = g_\theta$ as importance sampling density, such that $\mathbb{E}_g X_1 = 2 - \frac{1}{\log(2)}$.

c) Show that the random variable $\frac{1}{\log(2)} \log(X_1 + 1)$ has density function

$$h(x) = \log(2)\, 2^x\, \mathbb{I}\{0 \leq x \leq 1\}.$$

d) Write a Matlab program for estimating p for $n = 50$ by means of importance sampling with importance sampling density g based on 10000 realizations. Estimate p by the standard estimator and compare the sample variances of both estimators.