Methods of Monte Carlo Simulation II
Exercise Sheet 5
Deadline: June 5, 2014 at 10pm before the lecture

Exercise 1 (2+2)
Let \( \{X_n\}_{n \geq 0} \) be a Markov chain with transition matrix \( P = (P_{i,j})_{i,j=0,\ldots,l} \) where
\[
P_{i,j} = \begin{cases} \frac{l - i}{t} & \text{if } i < l \text{ and } j = i + 1 \\ \frac{1}{t} & \text{if } i > 0 \text{ and } j = i - 1 \\ 0 & \text{else.} \end{cases}
\]

a) Show that the stationary distribution \( \alpha = (\alpha_0, \ldots, \alpha_l) \) of \( \{X_n\}_{n \geq 0} \) is given by \( \alpha_i = 2^{-l} \binom{l}{i} \) for all \( i \in \{0, \ldots, l\} \).

b) Show that \( \{X_n\}_{n \geq 0} \) with initial distribution \( \alpha \) is reversible.

Exercise 2 (2+2+2)
Let \( \{X_n\}_{n \geq 0} \) be a random walk on the graph \( G = (V, E) \) with
\[
V = \{v_1, \ldots, v_8\} \\
E = \{(v_1, v_2), (v_1, v_3), (v_2, v_3), (v_2, v_4), (v_3, v_4), (v_3, v_7), (v_5, v_6), (v_6, v_7), (v_7, v_8)\}.
\]

a) Compute the stationary distribution \( \alpha \) of \( \{X_n\}_{n \geq 0} \).

b) Show that \( \{X_n\}_{n \geq 0} \) with initial distribution \( \alpha \) is reversible.

c) Consider the process \( \{X_n\}_{n \geq 0} \) with initial distribution \( \lambda = (0, 1/4, 0, 1/4, 0, 1/4, 0, 1/4) \). Write a Matlab program for estimating \( \mathbb{P}(X_n \in \{v_1, v_3, v_5, v_7\}) \) for each \( n \in \{1, 2, \ldots, 10\} \) based on 50000 realizations using the standard estimator.

Exercise 3 (3)
Let \( \{N_t\}_{t \geq 0} \) be a Poisson process with intensity \( \lambda \). Let \( \{\tilde{N}_t\}_{t \geq 0} \) be a process with independent and stationary increments, such that for all \( t \geq s \) it holds
\[
\mathbb{P}(\tilde{N}_t - \tilde{N}_s = k \mid N_t - N_s = n) = \binom{n}{k} p^{n-k} (1 - p)^k,
\]
for $n \geq k$ and
\[ \mathbb{P}(\tilde{N}_t - \tilde{N}_s = k \mid N_t - N_s = n) = 0, \]
for $n < k$. Moreover, let $\tilde{N}_0 = 0$. Show that $\{\tilde{N}_t\}_{t \geq 0}$ is a Poisson process with intensity $(1 - p)\lambda$.

**Exercise 4** \hspace{1cm} (3)

Let $\{N_t\}_{t \geq 0}$ be a Poisson process with intensity $\lambda > 0$. Calculate
\[ \mathbb{P}(N_5 - N_2 = 2 \mid N_4 = 4). \]

**Exercise 5** \hspace{1cm} (2+2+3)

Let $\{N_t\}_{t \geq 0}$ be a Poisson process with intensity $\lambda = 1$.

a) Write a Matlab program for simulating $\{N_t\}_{t \geq 0}$ using the definition of a Poisson process in terms of inter-arrival times. Simulate the process for $0 \leq t \leq 100$ and plot the realization of the path.

b) Write a Matlab program for estimating $\mathbb{E}N_t$ and $\text{Var}N_t$ for $t \in \{1, 2, \ldots, 10\}$. Use a sample size of 10000.

c) Write a Matlab program for estimating $\mathbb{P}(N_5 - N_2 = 2 \mid N_4 = 4)$. Do this by estimating $\mathbb{P}(N_5 - N_2 = 2, N_4 = 4)$ and $\mathbb{P}(N_4 = 4)$ separately using the standard Monte Carlo estimator. Use a sample size of 50000.