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# Methods of Monte Carlo Simulation II
# Exercise Sheet 5

Deadline: June 5, 2014 at 10pm before the lecture

**Exercise 1** (2+2)

Let $\{X_n\}_{n\geq 0}$ be a Markov chain with transition matrix $P = (P_{i,j})_{i,j=0,\ldots,l}$ where

$$P_{i,j} = \begin{cases} \frac{l-i}{l} & \text{if } i < l \text{ and } j = i+1 \\ \frac{i}{l} & \text{if } i > 0 \text{ and } j = i-1 \\ 0 & \text{else.} \end{cases}$$

a) Show that the stationary distribution $\alpha = (\alpha_0, \ldots, \alpha_l)$ of $\{X_n\}_{n\geq 0}$ is given by $\alpha_i = 2^{-l}\binom{l}{i}$ for all $i \in \{0, \ldots, l\}$.

b) Show that $\{X_n\}_{n\geq 0}$ with initial distribution $\alpha$ is reversible.

**Exercise 2** (2+2+2)

Let $\{X_n\}_{n\geq 0}$ be a random walk on the graph $G = (V, E)$ with

$$\begin{aligned} V =& \{v_1, \ldots v_8\} \\ E =& \{(v_1, v_2), (v_1, v_3), (v_2, v_3), (v_2, v_8), (v_3, v_4), (v_3, v_7), \\ & (v_3, v_8), (v_4, v_5), (v_4, v_6), (v_5, v_6), (v_6, v_7), (v_7, v_8)\}. \end{aligned}$$

a) Compute the stationary distribution $\alpha$ of $\{X_n\}_{n\geq 0}$.

b) Show that $\{X_n\}_{n\geq 0}$ with initial distribution $\alpha$ is reversible.

c) Consider the process $\{X_n\}_{n\geq 0}$ with initial distribution $\lambda = (0, 1/4, 0, 1/4, 0, 1/4, 0, 1/4)$. Write a Matlab program for estimating $\mathbb{P}(X_n \in \{v_1, v_3, v_5, v_7\})$ for each $n \in \{1, 2, \ldots, 10\}$ based on 50000 realizations using the standard estimator.

**Exercise 3** (3)

Let $\{N_t\}_{t\geq 0}$ be a Poisson process with intensity $\lambda$. Let $\{\widetilde{N}_t\}_{t\geq 0}$ be a process with independent and stationary increments, such that for all $t \geq s$ it holds

$$\mathbb{P}(\widetilde{N}_t - \widetilde{N}_s = k \mid N_t - N_s = n) = \binom{n}{k} p^{n-k}(1-p)^k,$$

for $n \geq k$ and

$$\mathbb{P}(\widetilde{N}_t - \widetilde{N}_s = k \mid N_t - N_s = n) = 0,$$

for $n < k$. Moreover, let $\widetilde{N}_0 = 0$. Show that $\{\widetilde{N}_t\}_{t\geq 0}$ is a Poisson process with intensity $(1-p)\lambda$.

**Exercise 4** (3)

Let $\{N_t\}_{t\geq 0}$ be a Poisson process with intensity $\lambda > 0$. Calculate

$$\mathbb{P}(N_5 - N_2 = 2 \mid N_4 = 4).$$

**Exercise 5** (2+2+3)

Let $\{N_t\}_{t\geq 0}$ be a Poisson process with intensity $\lambda = 1$.

a) Write a Matlab program for simulating $\{N_t\}_{t\geq 0}$ using the definition of a Poisson process in terms of inter-arrival times. Simulate the process for $0 \leq t \leq 100$ and plot the realization of the path.

b) Write a Matlab program for estimating $\mathbb{E}N_t$ and $\operatorname{Var} N_t$ for $t \in \{1, 2, \dots, 10\}$. Use a sample size of 10000.

c) Write a Matlab program for estimating $\mathbb{P}(N_5 - N_2 = 2 \mid N_4 = 4)$. Do this by estimating $\mathbb{P}(N_5 - N_2 = 2, N_4 = 4)$ and $\mathbb{P}(N_4 = 4)$ separately using the standard Monte Carlo estimator. Use a sample size of 50000.