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Methods of Monte Carlo Simulation II
Solution to Exercise Sheet 1

Exercise 1 (3)

Consider two gamblers, A and B, playing a game where A wins with probability $q \in (0,1)$ and B wins with probability $1-q$. Gambler A starts with s_A Euro and B starts with s_B Euro. The gambler who wins gets one Euro from his opponent. They repeat the game infinitely many times. Suppose that negative capital of one of the gamblers is possible.
Let $\{X_n\}_{n\geq 0}$ be the stochastic process where X_n denotes the capital in Euro of gambler A after n games. Show that $\{X_n\}_{n\geq 0}$ is a random walk.

Solution:

According to Definition 1.2.2 it is to show that $\{X_n\}_{n\geq 0}$ satisfies the recurrence

$$X_{n+1} = X_n + (2Y_{n+1} - 1)\,, \tag{1}$$

for each $n \geq 0$, where $\{Y_n\}_{n\geq 1}$ is a Bernoulli process. Since Gambler A wins with probability q, the process $\{X_n\}_{n\geq 0}$ increases in one step by one with probability q. In the other case $\{X_n\}_{n\geq 0}$ decreases by one. This occurs with probability $1-q$. Thus it holds

$$X_{n+1} = X_n + Z_{n+1} \tag{2}$$

for each $n \geq 0$, where $\{Z_n\}_{n\geq 1}$ is a sequence of i.i.d. random variables with $\mathbb{P}(Z_1 = -1) = 1-q$ and $\mathbb{P}(Z_1 = 1) = q$. Now we define the process $\{Y_n\}_{n\geq 1}$ by

$$Y_n = 0.5 Z_n + 0.5, \tag{3}$$

for each $n \geq 1$. Because $\{Z_n\}_{n\geq 1}$ is i.i.d., the sequence $\{Y_n\}_{n\geq 1}$ is i.i.d., too. Moreover, it holds $\mathbb{P}(Y_1 = 0) = \mathbb{P}(0.5Z_1 + 0.5 = 0) = \mathbb{P}(Z_1 = -1) = 1-q$ and $\mathbb{P}(Y_1 = 1) = \mathbb{P}(0.5Z_1 + 0.5 = 1) = \mathbb{P}(Z_1 = 1) = q$. Thus $\{Y_n\}_{n\geq 1}$ is a Bernoulli process. By (3) we can substitute Z_{n+1} in (2) by $2Y_{n+1} - 1$ which leads to (1). Thus $\{X_n\}_{n\geq 0}$ is a random walk which is uniquely defined by the initial condition $X_0 = s_A$.

Exercise 2 (3+3+4)

Consider the case of Exercise 1 with $q = 0.6, s_A = 6, s_B = 4$ and let $\{X_n\}_{n\geq 0}$ be defined as above.

a) Write a Matlab program for simulating the first N values of $\{X_n\}_{n\geq 0}$, i.e. $X_1, \ldots, X_N$. Plot one realization for each $N \in \{10, 100, 10000\}$.

b) Let $\tau = \inf\{n \geq 0 : X_n \in \{0, s_A + s_B\}\}$ be the random number of games after which the game ends if negative capital is not possible. Write a Matlab program for estimating $\mathbb{P}(\tau = 8)$ based on 1000 realizations of $\{X_n\}_{n\geq 0}$.

Hint: For estimating $\mathbb{P}(\tau = 8)$ simulate 1000 realizations $\{X_n^{(1)}\}_{n\geq 0}, \ldots, \{X_n^{(1000)}\}_{n\geq 0}$ of $\{X_n\}_{n\geq 0}$ and compute the corresponding values of τ denoted by $\tau^{(1)}, \ldots, \tau^{(1000)}$. Then, estimate the probability $\mathbb{P}(\tau = 8)$ by

$$\widehat{p}_8 = \frac{1}{1000} \sum_{i=1}^{1000} \mathbb{I}\left(\tau^{(i)} = 8\right).$$

c) Calculate $\mathbb{P}(\inf\{n \geq 0 : X_n \geq 8\} \leq 4)$ and write a Matlab programm to estimate $\mathbb{P}(\inf\{n \geq 0 : X_n \geq 8\} \leq 4)$ based on 1000 realizations of $\{X_n\}_{n\geq 0}$. Proceed analogously to part b).

Solution: a) The function which simulates the first N values of a random walk with parameter q and initial condition $X_0 = x_0$:

```
function X = RandomWalk(N,q,x_0)

% Simulating the Bernoulli process Y
Y = (rand(N,1) <= q);

% Generating the random walk
X = x_0 + cumsum((2*Y - 1));

end
```

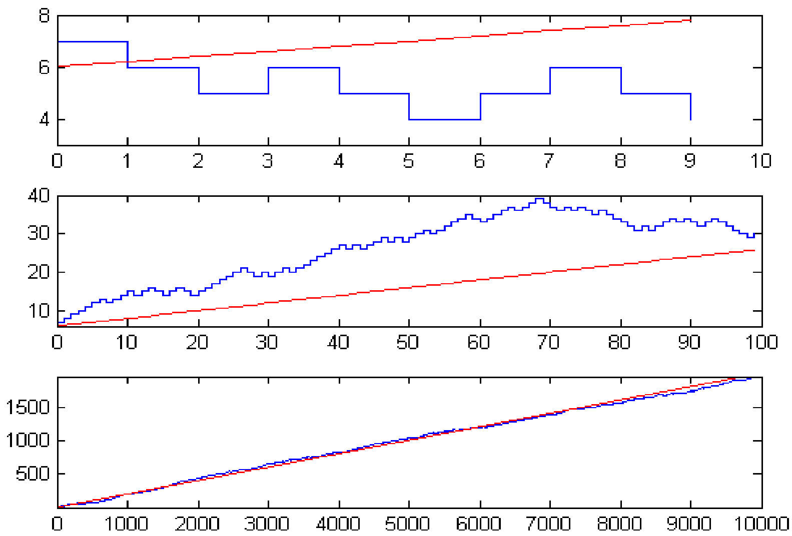

Figure 1: Plots of realizations. From to top to bottom: N=10, N=100, N=10000. The red line represents $\mathbb{E}X_n = 6 + 0.2\,n$.

The main program for simulating $\{X_n\}_{n\geq 0}$ with $N \in \{10, 100, 10000\}, X_0 = 6$ and $q = 0.6$:

```
% Exercise 2
% a)
close all;
clf;
N = [10 100 10000];

for i = 1:3
    % Simulate and plot a random walk
    X = RandomWalk(N(i), 0.5, 6);
    subplot(3,1,i);
    stairs(0:N(i)-1,X);

    % The line of the expected values is added to the plot
    hold on;
    t = 0:N(i)-1;
    plot(t, 6+0.2*t, 'red');

    axis([0 N(i) min(X)-1 max(X)+1]);
end
```

b) The program for estimating $\widehat{p_8}$:

```
% b)
tau = zeros(1000,1);

for i=1:1000
    cnt = 0;
    X = 6;
    % Simulating the random walk until X_n=0 or X_n=s_A+s_B=10
    while(X ~= 10 && X ~= 0)
        X = X + 2*(rand <= 0.6) - 1;
        cnt = cnt + 1;
    end
    tau(i) = cnt;
end

pHat = sum((tau == 8))/1000;
fprintf('The estimated value is %f\n', pHat);
```

Output:

```
The estimated value is 0.117000
```

With 100000 realizations of $\{X_n\}_{n\geq 0}$ the variance of the estimator is reduced and 0.11 was obtained as estimated value for p_8.

c) Using Lemma 1.2.5 and the result about the distribution of X_n from Section 1.2.4 we get

$$\begin{aligned}\mathbb{P}(\inf\{n \geq 0 : X_n \geq 8\} \leq 4) &= \mathbb{P}(\inf\{n \geq 0 : X_n - 6 \in \{2, 3, \ldots\}\} \leq 4) \\ &= \frac{2}{2}\,\mathbb{P}(X_2 - 6 = 2) + \frac{2}{4}\,\mathbb{P}(X_4 - 6 = 2) \\ &= \binom{2}{2} 0.6^2 + 0.5 \binom{4}{3} 0.6^3 \cdot 0.4 \\ &= 0.36 + 0.1728 = 0.5328.\end{aligned}$$

By the following Matlab programm $\mathbb{P}(\inf\{n \geq 0 : X_n \geq 8\} \leq 4)$ is estimated:

```
% c)
tau = zeros(1000,1);

for i=1:1000
    cnt = 0;
    X = 6;
    % Simulating the random walk until X_n=8
    while(X ~= 8)
        X = X + 2*(rand <= 0.6) - 1;
        cnt = cnt + 1;
    end
    tau(i) = cnt;
end

pHat = sum((tau <= 4))/1000;
fprintf('The estimated value is %f\n', pHat);
```

Output:

```
The estimated value is 0.540000
```