Problem 1 (6 points)
The Pareto distribution as well as the Weibull distribution are important in insurance mathematics. The density of a Pareto (Par(\(\alpha, c\))) distributed random variable with parameters \(\alpha, c > 0\) is given by

\[
f(x) = \frac{\alpha}{c} \left(\frac{c}{x}\right)^{\alpha+1} \mathbb{I}(x \in (c, \infty)).
\]

The density of a Weibull (W(\(r, c\))) distributed random variable with parameters \(r, c > 0\) is given by

\[
f(x) = rce^{-c(r-1)x}e^{-cx} \mathbb{I}(x \in [0, \infty)).
\]

(a) Compute the expected value and the variance of a Pareto distributed random variable with parameters \(\alpha > 2, c > 0\).

(b) Compute the distribution function of a Pareto distributed random variable with parameters \(\alpha, c > 0\).

(c) Compute the expected value and the variance of a Weibull distributed random variable with parameters \(r, c > 0\).

Problem 2 (4 points)
Let the duration \(T\) of a fire be an exponential distributed random variable with parameter \(\lambda > 0\). The damage \(l(t)\), caused by a fire of length \(t\), is given by \(l(t) = ae^{bt}\) with \(a, b > 0\). Show that the random variable \(l(T)\) is Pareto distributed with parameters \(\lambda b\) and \(a\).

Problem 3 (6 points)
(a) Let \(X\) be geometric distributed with parameter \(p \in (0, 1)\), that is \(\mathbb{P}[X = n] = p(1 - p)^n, n \in \mathbb{N}_0\). Show that for all \(i, j \in \mathbb{N}_0\)

\[
\mathbb{P}[X \leq i + j | X \geq j] = \mathbb{P}[X \leq i].
\]

(b) Let \(Y\) be exponential distributed with parameter \(\lambda > 0\). Show that for all \(t, s \geq 0\)

\[
\mathbb{P}[Y > t + s | Y > t] = \mathbb{P}[Y > s].
\]
(c) Let $Z_1 \sim \text{Exp}(\beta)$, $Z_2 \sim \text{Exp}(\gamma)$, $\beta, \gamma > 0$ be independent. Show that the minimum of these two random variables is also exponential distributed.

**Problem 4** (4 points)

(a) Let $X$ be Poisson distributed with parameter $\lambda > 0$. Calculate the moment-generating function $m_X(t) = \mathbb{E}[e^{tX}], t \in \mathbb{R}$.

(b) Let $X_1, \ldots, X_n, n \geq 2$, be i.i.d. random variables with $X_1 \sim \text{Poi}(\lambda), \lambda > 0$. Calculate the moment generating function of the sum $S_n = \sum_{i=1}^{n} X_i$. 