Problem 1 (6 credits)

(a) Let \( F_1, \ldots, F_n, n \geq 2 \) be distribution functions with moment generating functions \( \hat{m}_1, \ldots, \hat{m}_n \). Then, the convex combination \( F = \sum_{k=1}^{n} \mu_k F_k \), where \( \mu_k \geq 0 \), \( \sum_{k=1}^{n} \mu_k = 1 \) is also a distribution function. Let \( Y \) be a random variable with distribution function \( F \). Calculate the moment generating function \( \hat{m}_Y \) of \( Y \).

(b) The cumulant function of a random variable \( X \) is given by
\[
\log \hat{m}_X(r) = \log \mathbb{E}[e^{rX}].
\]
Assume that \( \hat{m}_X \) is finite for \((-r_0, r_0)\) with \( r_0 > 0 \). Calculate the first three derivatives of the cumulant function at \( r = 0 \) and show that it is convex on \((-r_0, r_0)\). You get 4 extra credits if you show that in this problem the derivative and the expected value in the moment generating function can be switched.

Problem 2 (5 credits)

The generating function \( \hat{g}_Y \) of a random variable \( Y : \Omega \rightarrow \mathbb{N} \) is given by
\[
\hat{g}_Y(u) = \mathbb{E}[u^Y], \text{ for } u \in [-1, 1].
\]

(a) Let \( X \sim \text{Exp}(\lambda) \) with \( \lambda > 0 \). Compute the moment generating function \( \hat{m}_X \) of \( X \).

(b) Let \( Y \sim \text{Poi}(\lambda) \) with \( \lambda > 0 \). Compute the generating function \( \hat{g}_Y \) of \( Y \).

(c) Let \( Z \) be negative binomial distributed, i.e. \( Z \sim \text{NB}(\alpha, p) \) with
\[
\mathbb{P}(Z = k) = \binom{\alpha + k - 1}{k} p^k (1 - p)^{\alpha}, \quad \alpha > 0, 0 < p < 1, k = 0, 1, \ldots.
\]
Compute the generating function \( \hat{g}_Z \) of \( Z \).

Problem 3 (3 credits)

Derive the hazard rate function of a risk \( X \), where \( X \) is

(a) Pareto distributed with parameters \( \alpha, c > 0 \).

(b) Weibull distributed with parameters \( r, c > 0 \).

(c) geometrically distributed with parameter \( p \in (0, 1) \).
**Problem 4** (4 credits)
Show that if $X$ is a risk with a heavy tail, then
\[ \limsup_{x \to \infty} e^{sx} \bar{F}_X(x) = \infty, \quad \forall s > 0. \]

**Problem 5** (6 credits)
(a) Show that the Weibull distribution $W(r, c)$ with parameters $r, c > 0$ is light-tailed for $r \geq 1$ and heavy-tailed for $r \in (0, 1)$.
(b) Show that the Pareto distribution $\text{Par}(\alpha, c)$ with parameters $\alpha, c > 0$ is heavy-tailed.
(c) Show that the gamma distribution $\Gamma(a, \lambda)$ with parameters $a, \lambda > 0$ is light-tailed.