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Risk Theory

Exercise Sheet 2

Due to: May 13, 2014

Note: Please submit exercise sheets in couples!

Problem 1 (6 credits)

(a) Let $F_1, \dots, F_n, n \geq 2$ be distribution functions with moment generating functions $\hat{m}_1, \dots, \hat{m}_n$. Then, the convex combination $F = \sum_{k=1}^{n} \mu_k F_k$, where $\mu_k \geq 0$, $\sum_{k=1}^{n} \mu_k = 1$ is also a distribution function. Let Y be a random variable with distribution function F. Calculate the moment generating function $\hat{m}_Y$ of Y.

(b) The cumulant function of a random variable X is given by

$$\log \hat{m}_X(r) = \log \mathbb{E}[e^{rX}].$$

Assume that $\hat{m}_X$ is finite for $(-r_0, r_0)$ with $r_0 > 0$. Calculate the first three derivatives of the cumulant function at $r = 0$ and show that it is convex on $(-r_0, r_0)$. You get 4 extra credits if you show that in this problem the derivative and the expected value in the moment generating function can be switched.

Problem 2 (5 credits)

The generating function $\hat{g}_Y$ of a random variable $Y : \Omega \to \mathbb{N}$ is given by

$$\hat{g}_Y(u) = \mathbb{E}[u^Y], \text{ for } u \in [-1, 1].$$

(a) Let $X \sim \text{Exp}(\lambda)$ with $\lambda > 0$. Compute the moment generating function $\hat{m}_X$ of X.

(b) Let $Y \sim \text{Poi}(\lambda)$ with $\lambda > 0$. Compute the generating function $\hat{g}_Y$ of Y.

(c) Let Z be negative binomial distributed, i.e. $Z \sim \text{NB}(\alpha, p)$ with

$$\mathbb{P}(Z = k) = \binom{\alpha + k - 1}{k} p^k (1-p)^\alpha, \ \alpha > 0, 0 < p < 1, k = 0, 1, \dots.$$

Compute the generating function $\hat{g}_Z$ of Z.

Problem 3 (3 credits)

Derive the hazard rate function of a risk X, where X is

(a) Pareto distributed with parameters $\alpha, c > 0$.

(b) Weibull distributed with parameters $r, c > 0$.

(c) geometrically distributed with parameter $p \in (0, 1)$.

Problem 4 (4 credits)

Show that if X is a risk with a heavy tail, then

$$\limsup_{x\to\infty} e^{sx}\bar{F}_X(x) = \infty, \ \forall s > 0.$$

Problem 5 (6 credits)

(a) Show that the Weilbull distribution $W(r, c)$ with parameters $r, c > 0$ is light-tailed for $r \geq 1$ and heavy-tailed for $r \in (0, 1)$.

(b) Show that the Pareto distribution $\mathrm{Par}(\alpha, c)$ with parameters $\alpha, c > 0$ is heavy-tailed.

(c) Show that the gamma distribution $\Gamma(a, \lambda)$ with parameters $a, \lambda > 0$ is light-tailed.