Problem 1 (3 credits)
Show that a real-valued stochastic process \( \{X(t), t \geq 0\} \) with independent increments has stationary increments if for all \( t \geq 0 \) the distribution of the (one-dimensional) random variable \( X(t + h) - X(h) \) does not depend on \( h \geq 0 \).

Problem 2 (9 credits)
Let \( \{N(t), t \geq 0\} \) be a mixed Poisson process with mixing random variable \( \Lambda \).
(a) Show that for the generating function \( \hat{g}_{N(t)}(s) = \mathbb{E}s^{N(t)}, |s| < 1 \) of \( N(t) \) it holds that
\[
\hat{g}_{N(t)}(s) = \hat{I}_\Lambda(t(1-s)), \forall t \geq 0.
\]
(b) Show that
\[
\begin{align*}
(1) & \ E[N(t) \cdot (N(t) - 1) \cdot \ldots \cdot (N(t) - k + 1)] = t^k \mathbb{E}[\Lambda^k], \ k \in \mathbb{N}, \\
(2) & \ E[N(t)] = t \cdot \mathbb{E}[\Lambda], \\
(3) & \ \text{Var}(N(t)) = t^2 \text{Var}(\Lambda) + t \mathbb{E}[\Lambda].
\end{align*}
\]

Problem 3 (4 credits)
Let \( \{N(t), t \geq 0\} \) be a Pascal process with parameters \( a, b > 0 \).
(a) Show that
\[
\hat{g}_{N(t)}(s) = \left( \frac{b}{b + t(1-s)} \right)^a, s \in (-1, 1), t \geq 0.
\]
(b) Show that
\[
p_k(t) = \binom{a+k-1}{k} \left( \frac{t}{t+b} \right)^k \left( \frac{b}{t+b} \right)^a
\]
i.e. \( N(t) \) follows a negative binomial distribution with parameters \( a \) and \( \frac{t}{t+b} \).
Problem 4 (4 credits)
Let \( \{N(t), t \geq 0\} \) be a Delaporte process. The density of the mixing random variable is given by

\[
f_\Lambda(\lambda) = \frac{\eta^a (\lambda - b)^{a-1}}{\Gamma(a)} e^{-\eta(\lambda - b) \mathbb{I}(\lambda > b)}.
\]

(a) Show that

\[
p_0(t) = e^{-bt} \left( \frac{\eta}{\eta + t} \right)^a.
\]

(b) Show that

\[
\hat{g}_{N(t)}(s) = e^{-(1-s)tb} \left( \frac{\eta}{\eta + (1-s)t} \right)^a, |s| < 1.
\]

Problem 5 (4 credits)
Determine the expected value, the variance and the index of dispersion of the negative binomial distribution with parameters \( \alpha > 0 \) and \( p \in (0, 1) \).