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The multidimensional Itô Integral and the multidimensional Itô Formula

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Recall - Class of integrands for 1-dimensional Itô Integral

Let $\mathcal{V} = \mathcal{V}(S, T)$ be the class of functions

$$f(t, \omega) : [0, \infty) \times \Omega \to \mathbb{R}$$

such that

(i) $(t, \omega) \to f(t, \omega)$ is $\mathcal{B} \times \mathcal{F}$ - measurable

(ii) $f(t, \omega)$ is $\mathcal{F}_t$ - adapted

(iii) $\mathbb{E}\,[\int_S^T f(t, \omega)^2 dt] < \infty$

Extension of $\mathcal{V}$

(ii)' There exists a filtration $\mathcal{H} = (\mathcal{H}_t)_{t\geq 0}$ such that

a) B_t is a martingale w.r.t. $(\mathcal{H}_t)_{t\geq 0}$

b) $f(t,\omega)$ is $\mathcal{H}_t$ - adapted

Example

Let $B_t(\omega) = (B_1(t,\omega), \dots, B_n(t,\omega))$, $0 \le t \le T$ be n-dimensional Brownian motion and define

$$\mathcal{F}_t^{(n)} = \sigma\left(B_i(s,\cdot) : 1 \le i \le n,\ 0 \le s \le t\right).$$

Then $B_k(t,\omega)$ is a martingale w.r.t. $\mathcal{F}_t^{(n)}$. Hence we can choose $\mathcal{H}_t = \mathcal{F}_t^{(n)}$ and thus $\int_0^t f(s,\omega)\, dB_k(s,\omega)$ exists for $\mathcal{F}_t^{(n)}$ - adapted integrands f.

Definition - multidimensional Itô Integral

Let $B(t,\omega) = (B_1(t,\omega),\dots,B_n(t,\omega))$ be n-dimensional Brownian motion and $v = [v_{ij}(t,\omega)]$ be a $m \times n$ - matrix where each entry $v_{ij}(t,\omega)$ satisfies (i), (iii) and (ii)' w.r.t. some filtration $\mathcal{H} = (\mathcal{H}_t)_{t\geq 0}$. Then we define

$$\int_S^T v\, dB = \int_S^T \begin{pmatrix} v_{11} & \dots & v_{1n} \\ \vdots & & \vdots \\ v_{m1} & \dots & v_{mn} \end{pmatrix} \begin{pmatrix} dB_1 \\ \vdots \\ dB_n \end{pmatrix}$$

to be the $m \times 1$ - matrix whose i'th component is

$$\sum_{j=1}^{n} \int_S^T v_{ij}(s,\omega)\, dB_j(s,\omega).$$

Definition - multidimensional Itô processes

Let $B(t,\omega) = (B_1(t,\omega), \ldots, B_m(t,\omega))$ denote m-dimensional Brownian motion. If the processes $u_i(t,\omega)$ and $v_{ij}(t,\omega)$ satisfy the conditions given in the definition of the 1-dimensional Itô process for each $1 \leq i \leq n$, $1 \leq j \leq m$ then we can form n 1-dimensional Itô processes

$$\begin{aligned} dX_1 &= u_1\, dt + v_{11}\, dB_1 + \ldots + v_{1m}\, dB_m \\ \vdots &\quad\;\; \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ dX_n &= u_n\, dt + v_{n1}\, dB_1 + \ldots + v_{nm}\, dB_m \end{aligned}$$

Or, in matrix notation

$$dX(t) = u\,dt + v\,dB(t)$$

where

$$X(t) = \begin{pmatrix} X_1(t) \\ \vdots \\ X_n(t) \end{pmatrix}, u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, v = \begin{pmatrix} v_{11} & \dots & v_{1m} \\ \vdots & & \vdots \\ v_{n1} & \dots & v_{nm} \end{pmatrix}, dB(t) = \begin{pmatrix} dB_1(t) \\ \vdots \\ dB_m(t) \end{pmatrix}$$

Then $X(t)$ is called an n-dimensional Itô process.

Theorem - The general Itô formula

Let

$$X(t) = X(0) + \int_0^t u(s)\, ds + \int_0^t v(s)\, dB(s)$$

be an n-dimensional Itô process. Let $g(t,x) = (g_1(t,x), \ldots, g_p(t,x))\,, p \in \mathbb{N}$, be a C^2 map from $[0,\infty) \times \mathbb{R}^n$ into $\mathbb{R}^p$. Then the process

$$Y(t,\omega) = g(t, X(t))$$

is again an Itô process, whose k'th component, $k = 1, \ldots, p$, is given by

$$Y_k(t) = Y_k(0) + \int_0^t \left(\frac{\partial g_k}{\partial t}(s, X(s)) + \sum_{i=1}^{n} \frac{\partial g_k}{\partial x_i}(s, X(s)) u_i(s) \right.$$

$$\left. + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 g_k}{\partial x_i \partial x_j}(s, X(s)) v_i(s) v_j(s)^T \right) \, ds$$

$$+ \sum_{i=1}^{n} \int_0^t \frac{\partial g_k}{\partial x_i}(s, X(s)) v_i(s) \, dB(s)$$

with $v_i(s)$ the i'th row of v.

Examples

a) Let $B(t,\omega) = (B_1(t,\omega), \dots, B_n(t,\omega))$ be an n-dimensional Brownian motion, $n \geq 2$, and consider

$$R(t,\omega) = \left(B_1^2(t,\omega) + \cdots + B_n^2(t,\omega)\right)^{1/2}$$

Then it follows with Itô's formula

$$R(t) = \sum_{i=1}^{n} \int_0^t \frac{B_i(s)}{R(s)}\, dB_i(s) + \int_0^t \frac{n-1}{2R(s)}\, ds$$

b) Let B_t be an 1-dimensional Brownian motion and $Y_t = 2 + t + e^{B_t}$. Then

$$Y_t = 3 + \int_0^t (1 + e^{B_s})\, ds + \int_0^t e^{B_s}\, dB_s$$

b) Let $B(t,\omega) = (B_1(t,\omega), B_2(t,\omega))$ be a 2-dimensional Brownian motion and $Y_t = B_1^2(t) + B_2^2(t)$. Then

$$Y_t = \int_0^t 2\, ds + \int_0^t 2B_1(s)\, dB_1(s) + \int_0^t 2B_2(s)\, dB_2(s)$$

d) With Itô's formula it holds that

$$\int_0^t B_s^2 \, dB_s = \frac{1}{3} B_t^3 - \int_0^t B_s \, ds$$

d) Let B_t be an 1-dimensional Brownian motion. Define

$$\beta_k = \mathbb{E}[B_t^k] \; ; \quad k = 0, 1, 2 \ldots ; \quad t \geq 0.$$

Use Itô's formula to prove that

$$\beta_k = \frac{1}{2} k(k-1) \int_0^t \beta_{k-2}(s) \, ds \; ; \quad k \geq 2.$$

Integration by parts

Let X_t, Y_t be two 1-dimensional Itô processes, i.e.,

$$X_t = X_0 + \int_0^t u_X(s)\, ds + \int_0^t v_X(s)\, dB_s$$

$$Y_t = Y_0 + \int_0^t u_Y(s)\, ds + \int_0^t v_Y(s)\, dB_s$$

Then it holds

$$\begin{aligned} X_t Y_t =& X_0 Y_0 + \int_0^t (X_s u_Y(s) + Y_s u_X(s) + v_X(s) v_Y(s))\; ds \\ &+ \int_0^t X_s v_Y(s) + Y_s v_X(s)\, dB_s \end{aligned}$$

Exponential martingales

Suppose $\theta(t,\omega) = (\theta_1(t,\omega),\ldots,\theta_n(t,\omega))$ with $\theta_k(t,\omega) \in \mathcal{V}(0,T)\,\forall k = 1,\ldots,n$, where $T \leq \infty$. Define

$$Z_t = exp\left\{\int_0^t \theta(s,\omega)\;dB(s) - \frac{1}{2}\int_0^t \theta(s,\omega)^T \cdot \theta(s,\omega)\;ds\right\}, \quad 0 \leq t \leq T$$

where B(s) is an n-dimensional Brownian motion. Then it holds

a)

$$Z_t = 1 + \int_0^t Z_s\theta(s,\omega)\;dB(s)$$

b) Z_t is a martingale for $t \leq T$, provided that

$$Z_t\theta_k(t,\omega) \in \mathcal{V}(0,T)\;\forall k = 1,\ldots,n.$$

References

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- Karatzas, I., Shreve, S.E. (1998) *Brownian Motion and Stochastic Calculus*, Second Edition, Springer-Verlag, New York