The construction of the Itô integral.

Mathias Thanner | May 2015 | Seminar on Stochastic Geometry and its applications

Definition 1

Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration on the probability space $(\Omega, \mathcal{F}, P)$ and $S, T \in \mathbb{R}_{\geq 0}$. Define $\mathcal{V} = \mathcal{V}(S, T)$ as the class of functions

$$f(t, \omega) : [S, T] \times \Omega \to \mathbb{R}$$

such that

(i) $(t, \omega) \mapsto f(t, \omega)$ is $\mathfrak{B} \times \mathcal{F}$-measurable, where $\mathfrak{B}$ denotes the Borel σ-algebra on $[S, T]$,

(ii) $f(t, \omega)$ is $\mathcal{F}_t$-adapted,

(iii) $E\left[\int_S^T f(t, \omega)^2 dt\right] < \infty$.

For random functions $f \in \mathcal{V}$ we will show how to define the Itô integral

$$\mathcal{I}[f](\omega) = \int_S^T f(t, \omega) dB_t(\omega),$$

where B_t is a 1-dimensional Brownian motion and $S, T \in \mathbb{R}_{\geq 0}$.

Idea of construction

1. Define $\mathcal{I}[\phi]$ for a simple class of functions ϕ.
2. Show that each $f \in \mathcal{V}$ can be approximated (in an appropriate sense) by such ϕ's.
3. Use this to define $\int f dB$ as the limit of $\int \phi dB$ as $\phi \to f$.

Definition 2 - Elementary functions

A function $\phi \in \mathcal{V}$ is called elementary if it has the form

$$\phi(t, \omega) = \sum_{j \geq 0} e_j(\omega) \cdot \mathbb{I}_{[t_j, t_{j+1})}(t),$$

where $e_j(\omega) : \Omega \to \mathbb{R}$ for all $j \in \mathbb{N}_0$ and $t_0 < t_1 < t_2 < \ldots$ is a partition of the interval considered.

Remark 1

Note that since $\phi \in \mathcal{V}$ each function e_j must be $\mathcal{F}_{t_j}$-measureable.

Definition 3 - Integral

For elementary functions $\phi(t, \omega)$ the integral is defined in the following way

$$\int_S^T \phi(t, \omega) dB_t(\omega) := \sum_{j=0}^{n-1} e_j(\omega) \left[B_{t_{j+1}} - B_{t_j} \right] (\omega).$$

where $n \in \mathbb{N}$ and $S = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$ is a partition of the interval $[S, T] \subset \mathbb{R}_{\geq 0}$.

Remark 2

Each function $e_j(\omega)$ is independent of $\left[B_{t_{j+1}} - B_{t_j} \right] (\omega)$, since as a result of the independent increments of the Brownian motion $\left[B_{t_{j+1}} - B_{t_j} \right] (\omega) \perp \mathcal{F}_{t_j}$ and by Remark 1 $e_j(\omega)$ is $\mathcal{F}_{t_j}$-measurable.

Lemma - The Itô isometry

If $\phi(t,\omega)$ is bounded and elementary then

$$E\left[\left(\int_S^T \phi(t,\omega)dB_t(\omega)\right)^2\right] = E\left[\int_S^T \phi(t,\omega)^2 dt\right]. \tag{1}$$

Remark 3

$E\left[\int_S^T \phi(t,\omega)dB_t(\omega)\right] = E\left[\sum_{j=0}^{n-1} e_j(\omega)\left[B_{t_{j+1}} - B_{t_j}\right](\omega)\right] = 0$, since $e_j(\omega) \perp \left[B_{t_{j+1}} - B_{t_j}\right](\omega)$ and $E\left[\left[B_{t_{j+1}} - B_{t_j}\right](\omega)\right] = 0,\ \forall j \in \{0,1,\ldots,n-1\}$.

Approximation - Step 1

Let $g \in \mathcal{V}$ be bounded and $g(\cdot, \omega)$ continuous for each ω. Then there exist elementary functions $\phi_n \in \mathcal{V}$ such that

$$E\left[\int_S^T (g - \phi_n)^2 \, dt\right] \to 0 \quad \text{as } n \to \infty.$$

Approximation - Step 2

Let $h \in \mathcal{V}$ be bounded. Then there exist bounded functions $g_n \in \mathcal{V}$ such that $g_n(\cdot, \omega)$ is continuous for all ω and n, and

$$E\left[\int_S^T (h - g_n)^2 \, dt\right] \to 0 \quad \text{as } n \to \infty.$$

Approximation - Step 3

Let $f \in \mathcal{V}$. Then there exists a sequence $\{h_n\} \subset \mathcal{V}$ such that h_n is bounded for each n and

$$E\left[\int_S^T (f - h_n)^2 \, dt\right] \to 0 \quad \text{as } n \to \infty.$$

If $f \in \mathcal{V}$ we choose, by Steps 1-3, elementary functions $\phi_n \in \mathcal{V}$ such that

$$E\left[\int_S^T (f - \phi_n)^2 \, dt\right] \to 0 \quad \text{as } n \to \infty.$$

Then we define

$$\mathcal{I}[f](\omega) := \int_S^T f(t,\omega) dB_t(\omega) := \lim_{n\to\infty} \int_S^T \phi_n(t,\omega) dB_t(\omega).$$

Remark 4

The limit exists as an element of $L^2(P)$, since $\{\int_S^T \phi_n(t,\omega) dB_t(\omega)\}$ forms a Cauchy sequence in $L^2(P)$ by (1).

Definition 4 - The Itô integral

Let $f \in \mathcal{V}(S, T)$. Then the Itô integral of f (from S to T) is defined by

$$\int_S^T f(t,\omega) dB_t(\omega) = \lim_{n\to\infty} \int_S^T \phi_n(t,\omega) dB_t(\omega) \quad \text{(limit in } L^2(P)) \tag{2}$$

where $\{\phi_n\}$ is a sequence of elementary functions such that

$$E\left[\int_S^T (f(t,\omega) - \phi_n(t,\omega))^2 \, dt\right] \to 0 \quad \text{as } n \to \infty. \tag{3}$$

Remark 5

(i) Note that such a sequence $\{\phi_n\}$ satisfying (3) exists by Steps 1-3 from above.

(ii) Moveover, by (1) the limit in (2) exists and does not depend on the actual choice of $\{\phi_n\}$, as long as (3) holds.

Corollary 1 - The Itô isometry

$$E\left[\left(\int_S^T f(t,\omega)dB_t(\omega)\right)^2\right] = E\left[\int_S^T f(t,\omega)^2 dt\right] \quad \text{for all } f \in \mathcal{V}(S,T).$$

Corollary 2

If $f(t,\omega) \in \mathcal{V}(S,T)$ and $f_n(t,\omega) \in \mathcal{V}(S,T)$ for $n = 1, 2, \ldots$ and $E\left[\int_S^T (f_n(t,\omega) - f(t,\omega))^2 \, dt\right] \to 0$ as $n \to \infty$, then

$$\int_S^T f_n(t,\omega) dB_t(\omega) \to \int_S^T f(t,\omega) dB_t(\omega) \quad \text{in } L^2(P) \text{ as } n \to \infty.$$

Example

Assume $B_0 = 0$. Then

$$\int_0^t B_s dB_s = \frac{1}{2} B_t^2 - \frac{1}{2} t.$$

Proof of the Example - 1

Let $0 = t_0 < t_1 < \ldots t_{n-1} < t_n = t$ be a partition of the interval $[0, t]$ for some $t > 0$ and $n \in \mathbb{N}$.
Define $\phi_n(s, \omega) = \sum_{j=0}^{n-1} B_j(\omega) \cdot \mathbb{I}_{[t_j, t_{j+1})}(s)$, where $B_j = B_{t_j}$.
First we have to show that the condition in Corollary 2 is satisfied:

$$E\left[\int_0^t (\phi_n - B_s)^2 \, ds\right] = E\left[\sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \underbrace{(B_j - B_s)^2}_{\geq 0} ds\right] = \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \overbrace{E\left[(\underbrace{B_s - B_j}_{\sim N(0, s-t_j)})^2\right]}^{=Var(B_s - B_j)} ds$$

$$= \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (s - t_j) \, ds = \sum_{j=0}^{n-1} \left[\frac{1}{2} s^2 - s \cdot t_j\right]_{s=t_j}^{s=t_{j+1}} = \sum_{j=0}^{n-1} \left(\frac{1}{2} t_{j+1}^2 - t_j \cdot t_{j+1} \underbrace{- \frac{1}{2} t_j^2 + t_j^2}_{= \frac{1}{2} t_j^2}\right)$$

$$= \sum_{j=0}^{n-1} \frac{1}{2} (\underbrace{t_{j+1} - t_j}_{=: \Delta t_j})^2 \to 0 \quad \text{as } \Delta t_j \to 0.$$

Proof of the Example - 2

By Corollary 2 we get that

$$\int_0^t B_s dB_s = \lim_{\Delta t_j \to 0} \int_0^t \phi_n dB_s = \lim_{\Delta t_j \to 0} \sum_{j=0}^{n-1} B_j \Delta B_j.$$

Moreover

$$\Delta(B_j^2) = B_{j+1}^2 - B_j^2 = (B_{j+1} - B_j)^2 + 2B_j (B_{j+1} - B_j) = (\Delta B_j)^2 + 2B_j \Delta B_j.$$

Then, since $B_0 = 0$,

$$B_t^2 = B_{t_n}^2 = B_{t_n}^2 - B_{t_{n-1}}^2 + B_{t_{n-1}}^2 + \cdots - B_{t_1}^2 + B_{t_1}^2 - B_{t_0}^2$$
$$= \sum_{j=0}^{n-1} \Delta(B_j^2) = \sum_{j=0}^{n-1} (\Delta B_j)^2 + 2 \sum_{j=0}^{n-1} B_j \Delta B_j.$$

Proof of the Example - 3

$$\Rightarrow \int_0^t B_s dB_s = \lim_{\Delta t_j \to 0} \sum_{j=0}^{n-1} B_j \Delta B_j = \frac{1}{2} B_t^2 - \frac{1}{2} \underbrace{\lim_{\Delta t_j \to 0} \sum_{j=0}^{n-1} (\Delta B_j)^2}_{= t \text{ in } L^2(P)} = \frac{1}{2} B_t^2 - \frac{1}{2} t.$$

Remark 6

The extra term $-\frac{1}{2}t$ shows that the Itô stochastic integral does not behave like ordinary integrals.

Literature

B. Øksendal. *Stochastic differential equations.* Springer, 2003.

Thank you for your attention!