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Seminar Martingale

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Conditional Expectation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $X : \Omega \to \mathbb{R}^n$ be a random variable, $\mathbb{E}[|X|] < \infty$ and $\mathcal{H} \subset \mathcal{F}$ *a* σ-algebra, then the *conditional expectation of* X *given* $\mathcal{H}$, denoted by $\mathbb{E}[X|\mathcal{H}]$, is defined as follows:

Defintion:

$\mathbb{E}[X|\mathcal{H}]$ *is the (a.s. unique) function from* Ω *to* $\mathbb{R}^n$ *satisfying:*

1. $\mathbb{E}[X|\mathcal{H}]$ *is* $\mathcal{H}$*-measurable*
2. $\int_H \mathbb{E}[X|\mathcal{H}] d\mathbb{P} = \int_H X \, d\mathbb{P}$ *for all* $H \in \mathcal{H}$

Example:

Let $\Omega = \{1,2,3,4,5,6\}$ be the set of numbers of a die, $\mathcal{P}(\Omega) = \mathcal{F}$ and $X : \Omega \to \mathbb{N}$ be the random variable with $X(\omega) = \omega$ (the number of the die). Now we hide the numbers 1 and 6 by covering them. Thus our observations get inaccurate and our new σ-algebra is $\mathcal{H} = \sigma(\{2\},\{3\},\{4\},\{5\},\{1,6\})$. So $\mathcal{H} \subset \mathcal{F}$.

What is happening to X? X is not measurable to $\mathcal{H}$. So we create an appropriate RV ($\mathbb{E}[X|\mathcal{H}]$) s.t.

1. $\mathbb{E}[X|\mathcal{H}]$ is $\mathcal{H}$-measurable
2. $\mathbb{E}[\mathbb{E}[X|\mathcal{H}] \cdot \mathcal{X}_H] = \mathbb{E}[X \cdot \mathcal{X}_H]$ for all $H \in \mathcal{H}$

We define

$$\mathbb{E}[X|\mathcal{H}](\omega) := X(\omega), \ \textit{for } \omega = 2,3,4,5$$

$$\mathbb{E}[X|\mathcal{H}](\omega) := \frac{1+6}{2} = 3.5, \ \textit{for } \omega = 1,6$$

Obviously $\mathbb{E}[X|\mathcal{H}]$ satisfies 1. and 2. .

Proof:

We want to show the existence and the a.s. uniqueness of $\mathbb{E}[X|\mathcal{H}]$. Let ν be the intergral of X over a set H:

$$\nu(H) := \int_H X \, d\mathbb{P} \textit{ for all } H \in \mathcal{H}$$

It is easy to see, that ν is a finite signed measure on $\mathcal{H}$. Furthermore, it is $\forall\ H \in \mathcal{H}$, if $\mathbb{P}(H) = 0$, then $\nu(H) = 0$. So ν is absolutely continuous w.r.t. $\mathbb{P}|\mathcal{H}$.

As $(\Omega, \mathcal{H}, \mathbb{P}|\mathcal{H})$ is a σ-finite space, we can apply the theorem of Radon-Nikodym, which says there is a $P|\mathcal{H}$-unique $\mathcal{H}$-measurable function F on Ω such that

$$\nu(H) := \int_H F \, d\mathbb{P} \textit{ for all } H \in \mathcal{H}$$

We define $\mathbb{E}[X|\mathcal{H}] := F$ and this function is unique w.r.t. to the measure $P|\mathcal{H}$.

Now we consider the most important properties of the conditional expectation:

Theorem:

Suppose $Y : \Omega \to \mathbb{R}^n$ *is another random variable with* $\mathbb{E}[|Y|] < \infty$ *and let* $a, b \in \mathbb{R}$. *Then*

a) $\mathbb{E}[aX + bY|\mathcal{H}] = a\mathbb{E}[X|\mathcal{H}] + b\mathbb{E}[Y|\mathcal{H}]$

b) $\mathbb{E}[\mathbb{E}[X|\mathcal{H}]] = \mathbb{E}[X]$

c) $\mathbb{E}[X|\mathcal{H}] = X$ *if* X *is* $\mathcal{H} - measurable$

d) $\mathbb{E}[X|\mathcal{H}] = \mathbb{E}[X]$ *if* X *is independent of* $\mathcal{H}$

e)$\mathbb{E}[Y \cdot X|\mathcal{H}] = Y \cdot \mathbb{E}[X|\mathcal{H}]$ *if* $X, Y \in \mathcal{L}^2$ *and* Y *is* $\mathcal{H} -$ *measurable, where* $\cdot$ *denotes the usual inner product in* $\mathbb{R}^n$

Proof:

b) Assume $H = \Omega \in \mathcal{H}$. Then

$$\mathbb{E}[\mathbb{E}[X|\mathcal{H}] \cdot \mathcal{X}_H] = \int_H \mathbb{E}[X|\mathcal{H}] d\mathbb{P} \overset{2.}{=} \int_H X \, d\mathbb{P} = \mathbb{E}[X]$$

c) As X is $\mathcal{H}$-measurable, X satisfies both 1. and 2. . Because of that, and the fact that $\mathbb{E}[X|\mathcal{H}]$ is a.s. unique, we conclude $X = \mathbb{E}[X|\mathcal{H}]$.

d) We show, that $\mathbb{E}[X]$ satisfies 1. and 2. . As $\mathbb{E}[X]$ is a constant, 1. is satisfied. If X is independent of $\mathcal{H}$ we have for $H \in \mathcal{H}$

$$\int_H \mathbb{E}[X] d\mathbb{P} = \mathbb{E}[X] \cdot \mathbb{P}[H] = \int_\Omega X \, d\mathbb{P} \cdot \int_\Omega \mathcal{X}_H d\mathbb{P}$$

$$= \int_\Omega X \cdot \mathcal{X}_H d\mathbb{P} = \int_H X \, d\mathbb{P}$$

e) We show that $Y \cdot \mathbb{E}[X|\mathcal{H}]$ satisfies 1. and 2. . As Y and $\mathbb{E}[X|\mathcal{H}]$ are both measurable w.r.t. $\mathcal{H}$, we conclude that the product is also $\mathcal{H}$-measurable. To show property 2., we first consider $Y = \mathcal{X}_G$ ($\mathcal{H}$-measurable) for some $G \in \mathcal{H}$.
Then for all $H \in \mathcal{H}$

$$\int_H Y \cdot \mathbb{E}[X|H] d\mathbb{P} = \int_{H \cap G} \mathbb{E}[X|H] d\mathbb{P} \overset{2.}{=} \int_{H \cap G} X \, d\mathbb{P} = \int_H YX \, d\mathbb{P}$$

Similarly, we obtain that the result is true if

$$Y := \sum_{j=1}^{m} c_j \mathcal{X}_{G_j}, \text{ where } G_j \in \mathcal{H}.$$

As we can approximate every $\mathcal{H}$-measurable RV Y by such simple functions, we proved the statement.

Theorem:
Let $\mathcal{G}$, $\mathcal{H}$ be σ-algebras such that $\mathcal{G} \subset \mathcal{H}$. Then

$$\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[\mathbb{E}[X|\mathcal{H}]|\mathcal{G}].$$

Proof:
If $G \in \mathcal{G}$ then $G \in \mathcal{H}$ and therefore

$$\mathbb{E}[\mathbb{E}[\mathbb{E}[X|\mathcal{H}]|\mathcal{G}] \cdot \mathcal{X}_G] = \int_G \mathbb{E}[X|\mathcal{H}] d\mathbb{P} = \int_G X \, d\mathbb{P}$$

Once again, 1. and 2. are satisfied.
Hence $\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[\mathbb{E}[X|\mathcal{H}]|\mathcal{G}]$ by uniqueness.

Theorem: (The Jensen inequality)
If $\phi : \mathbb{R} \to \mathbb{R}$ is convex and $\mathbb{E}[|\phi(X)|] < \infty$ then

$$\phi(\mathbb{E}[X|\mathcal{H}]) \leq \mathbb{E}[\phi(X)|\mathcal{H}]$$

Corollary:
(i) $|\mathbb{E}[X|\mathcal{H}]| \leq \mathbb{E}[|X||\mathcal{H}]$
(ii)$|\mathbb{E}[X|\mathcal{H}]|^2 \leq \mathbb{E}[|X|^2|\mathcal{H}]$
Proof:
(i) It is

$$|\mathbb{E}[X|\mathcal{H}]| = |\mathbb{E}[X^+ - X^-|\mathcal{H}]| = |\mathbb{E}[X^+|\mathcal{H}] - \mathbb{E}[X^-|\mathcal{H}]| \leq \mathbb{E}[X^+|\mathcal{H}] + \mathbb{E}[X^-|\mathcal{H}] = \mathbb{E}[|X||\mathcal{H}]$$

(ii) Define $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(x) := x^2$. Then ϕ is convex and we can apply the Jensen inequality on $\phi(\mathbb{E}[X|\mathcal{H}])$.

Corollary:

If $X_n \to X$ in L^2 then $\mathbb{E}[X_n|\mathcal{H}] \to \mathbb{E}[X|\mathcal{H}]$ in L^2.

Proof:

We have to show:

(1) $\mathbb{E}[X|\mathcal{H}], \mathbb{E}[X_n|\mathcal{H}] \in L^2 \ \forall n$

(2) $\lim_{n\to\infty} \mathbb{E}[(\mathbb{E}[X_n|\mathcal{H}] - \mathbb{E}[X|\mathcal{H}])^2] = 0$

It is

$$\mathbb{E}[(\mathbb{E}[X_n|\mathcal{H}])^2] = \mathbb{E}[|\mathbb{E}[X_n|\mathcal{H}]|^2] \overset{(ii)}{\leq} \mathbb{E}[\mathbb{E}[|X_n|^2|\mathcal{H}]] = \mathbb{E}[\mathbb{E}[X_n^2|\mathcal{H}]] = \mathbb{E}[X_n^2] \overset{X_n\in L^2}{<} \infty$$

So $\mathbb{E}[X_n|\mathcal{H}] \in L^2 \ \forall n$. Similarly we obtain that $\mathbb{E}[X|\mathcal{H}] \in L^2$.

To show (2), we first take a look on

$$\mathbb{E}[(\mathbb{E}[X_n|\mathcal{H}] - \mathbb{E}[X|\mathcal{H}])^2] = \mathbb{E}[(\mathbb{E}[X_n - X|\mathcal{H}])^2]$$
$$\leq \mathbb{E}[\mathbb{E}[(X_n - X)^2|\mathcal{H}]] = \mathbb{E}[(X_n - X)^2]$$

As n was arbitrary, we conclude:

$$\lim_{n\to\infty} \mathbb{E}[(\mathbb{E}[X_n|\mathcal{H}] - \mathbb{E}[X|\mathcal{H}])^2] \leq \lim_{n\to\infty} \mathbb{E}[(X_n - X)^2] = 0$$

It is $\mathbb{E}[(\mathbb{E}[X_n|\mathcal{H}] - \mathbb{E}[X|\mathcal{H}])^2] \geq 0$ and we follow

$$\lim_{n\to\infty} \mathbb{E}[(\mathbb{E}[X_n|\mathcal{H}] - \mathbb{E}[X|\mathcal{H}])^2] = 0$$

Martingales

Let $(\Omega, \mathcal{N}, \mathbb{P})$ be a probability space and let $\{\mathcal{N}_t\}_{t\geq 0} \subset \mathcal{N}$ be a filtration, i.e. $\{\mathcal{N}_t\}_{t\geq 0}$ is a family of increasing σ-algebras.

Definition:

A stochastic process $\{N_t : t \geq 0\}$ with $N_t : \Omega \to \mathbb{R}$ is adapted if N_t is $\mathcal{N}_t$-measurable $\forall t \geq 0$.

Defintion:

A stochastic process $\{N_t : t \geq 0\}$ is a martingale if

1. N_t is $\mathcal{N}_t$-adapted
2. $\mathbb{E}[|N_t|] < \infty$
3. $\mathbb{E}[N_s|\mathcal{N}_t] = N_t \ \forall 0 \leq t \leq s$

Definition:

$\{N_t : t \geq 0\}$ is called submartingale if 1. and 2. and

3(a). $\mathbb{E}[N_s|\mathcal{N}_t] \geq N_t \ \forall 0 \leq t < s$

X is called supermartingale if 1. and 2. and

3(b). $\mathbb{E}[N_s|\mathcal{N}_t] \leq N_t \ \forall 0 \leq t < s$

Example:

Brownian Motion $\{B(t) : t \geq 0\}$ is a martingale w.r.t. to the natural filtration $\mathcal{F}_t = \sigma\{B(s) : 0 \leq s \leq t\}$.

1. follows by definition and 2. is satisfied as $\mathbb{E}[B(t)] = 0$ for all $t \geq 0$. Furthermore, we have

$$\mathbb{E}[B(t)|\mathcal{F}_s] = \mathbb{E}[B(s) + (B(t) - B(s))|\mathcal{F}_s] =$$
$$\mathbb{E}[B(s)|\mathcal{F}_s] + \mathbb{E}\underbrace{[B(t) - B(s)|\mathcal{F}_s]}_{independent} = B(s) + \mathbb{E}\underbrace{[B(t) - B(s)]}_{N(0,t-s)} = B(s)$$

Example:

Assume $\{N_t : t \geq 0\}$ is a martingale and $\{|N_t| : t \geq 0\} \in L^1$.
Then $\{|N_t| : t \geq 0\}$ is a submartingale.
Proof: Obviously 1. and 2. are satisfied. To show 3(a)., we use the Jensen inequality.

$$\mathbb{E}[|N_s||\mathcal{N}_t] \geq |\mathbb{E}[N_s|\mathcal{N}_t]| = |N_t| \; \forall 0 \leq t < s$$

In conclusion, we obtain that a convex function of a martingale is a submartingale.

Example:

We consider $t \in \mathbb{N}_0$ (discrete time). A gambler wins 1\$ when a coin comes up heads and loses 1\$ when the coin comes up tails. Suppose now that the coin comes up heads with probability $p \leq \frac{1}{2}$. On average, the gambler loses money and his fortune over time is a supermartingale.

As in customary we will assume that each $\mathcal{N}_t$ contains all the null sets of $\mathcal{N}$, that $t \to N_t(\omega)$ is right continous for a.a. ω and that $\{\mathcal{N}_t\}$ is right continous, in the sense that $\mathcal{N}_t = \bigcap_{s>t} \mathcal{N}_s$ for all $t \geq 0$.

<u>*Theorem:*</u> *(Doob's Martingale convergence theorem 1)*
Let N_t be a right continuous supermartingale, $sup_{t>0}\mathbb{E}[N_t^-] < \infty$ with $N_t^- = max(-N_t, 0)$. Then

$$N(\omega) = \lim_{t\to\infty} N_t(\omega)$$

exists for a.a. ω and $\mathbb{E}[N^-] < \infty$.
If we assume that $N_t(\omega)$ is bounded in L^1 for all t, then $N(\omega)$ is finite a.s. .

Example:

Let us take a look on the harmonic series: We know that

$$\sum_{k=1}^{\infty} \frac{1}{k} = \infty$$

But what about the following series

$$\sum_{k=1}^{\infty} \frac{\xi_k}{k}$$

where ξ_k are independent and identically distributed RVs with $\mathbb{P}[\xi_k = +1] = \mathbb{P}[\xi_k = -1] = \frac{1}{2}$?

If we consider the case $\xi_1 = -1,\ \xi_2 = +1,\ \xi_3 = -1, \ldots$ we have $\sum_{k=1}^{\infty} \frac{\xi_k}{k} = -1 + \frac{1}{2} - \frac{1}{3} + \ldots < \infty$ because of the alternating series test.

So we assume: $\sum_{k=1}^{\infty} \frac{\xi_k}{k} < \infty$.

Proof:

Consider the partial sum $S_n := \sum_{k=1}^{n} \frac{\xi_k}{k}$ with $S_0 := 0$.

S_n is a martingale w.r.t. $\sigma(\xi_1, ...\xi_n)$:

1. S_n *is* $\sigma(\xi_1, ...\xi_n) =: \mathcal{F}_n$ *adapted*

3. $\mathbb{E}[S_n|\mathcal{F}_{n-1}] = \mathbb{E}[S_{n-1} + \xi_n|\mathcal{F}_{n-1}] =$

$$\mathbb{E}[\underbrace{S_{n-1}|\mathcal{F}_{n-1}}_{\textit{measurable}}] + \mathbb{E}[\underbrace{\xi_n|\mathcal{F}_{n-1}}_{\textit{independent}}] = S_{n-1} + \mathbb{E}[\xi_n] \overset{\mathbb{E}[\xi_k]=0}{=} S_{n-1}$$

For property 2., we need to show $\mathbb{E}[|S_n|] < \infty \; \forall n \in \mathbb{N}$.

Therefore we prove that S_n is bounded in L^2. Then S_n is bounded in L^1 and property 2. is fulfilled.

Consider:

$$\mathbb{E}[S_n^2] = \mathbb{E}[S_n^2] - (\mathbb{E}[S_n])^2 = Var(S_n) \overset{\xi_k \, indep.}{=} \sum_{k=1}^{n} Var[\frac{\xi_k}{k}]$$

$$\overset{Var[\xi_k]=1}{=} \sum_{k=1}^{n} \frac{1}{k^2} < \frac{\pi^2}{6} \; \forall n \in \mathbb{N}$$

As $\lim_{n\to\infty} \mathbb{E}[S_n^2] = \frac{\pi^2}{6}$, it also follows that $\sup_{n\in\mathbb{N}} \mathbb{E}[S_n^2] \leq \frac{\pi^2}{6}$.
So $\{S_n\}_{n\in\mathbb{N}}$ is bounded in L^2
$\Rightarrow \{S_n\}_{n\in\mathbb{N}}$ is bounded in L^1
We conclude that 2. property is fullfilled. So S_n is a martingale and thus also a supermartingale.

Because of the fact, that $\{S_n\}_{n\in\mathbb{N}}$ is bounded in L^1, we can apply Doob's martingale covergence theorem $\Rightarrow \lim_{n\to\infty} S_n$ exists in $\mathbb{R}$ a.s.
Notice that if ξ_k are positive RV with $Var(\xi_k) = \sigma$ for all $k \in \mathbb{N}$, we can achieve the same result.

Definition:

A family C of RV N_t on a probability space is uniformly integrable (UI) if

$$\lim_{K\to\infty}(\sup_{t\geq 0}\mathbb{E}[|N_t|\mathcal{X}_{|N_t|>K}]) = 0$$

Theorem:

If a family C is bounded in $L^p (p > 1)$, then it is UI.

Theorem: (Doob's martingale convergence theorem 2)
Let N_t be a right-continuous supermartingale. Then the following are equivalent:

1. $\{N_t\}_{t \geq 0}$ *is uniformly integrable*
2. $\exists$ *RV* $N \in L_1$ *s.t.* $N_t \overset{a.e., L^1}{\rightarrow} N$.

Example:
We already know that $\lim_{n\to\infty} S_n(\omega) = S(\omega)$, where S is a finite RV. As $\{S_n\}_{n\in\mathbb{N}}$ is L^2-bounded, it is UI. With Doob's martingale convergence theorem 2 we also obtain that $\lim_{n\to\infty} \mathbb{E}[|S_n - S|] = 0$.

References

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Thank you for your attention