The Itô Formula
Proof and Applications

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Overview

Introduction

Using the definition is not very useful to evaluate a given integral (similar to Riemann integrals).

Remember, in order to show that

$$\int_0^t B_s dB_s = \frac{1}{2} B_t^2 - \frac{1}{2} t$$

we had to find elementary functions f_n such that

$$\int_0^t f_n(t, \omega) dB_t(\omega) \to \int_0^t B_s dB_s.$$

Definition

Let $\mathcal{V} := \mathcal{V}(S, T)$ be the class of functions

$$f(t,\omega) : [0,\infty) \times \Omega \to \mathbb{R}$$

1. $(t,\omega) \to f(t,\omega)$ is $\mathcal{B} \times \mathcal{F}$-measurable, where $\mathcal{B}$ denotes the Borel σ-algebra on $[0,\infty)$.
2. $f(t,\omega)$ is $\mathcal{F}_t$-adapted.
3. $\mathbb{E}\left[\int_S^T f(t,\omega)^2 dt\right] < \infty$.

Definition

Let B_t be the 1-dimensional Brownian motion on $(\Omega, \mathcal{F}, P)$. An *Itô process* is a stochastic process X_t on $(\Omega, \mathcal{F}, P)$ of the form

$$X_t = X_0 + \int_0^t u(s,\omega)ds + \int_0^t v(s,\omega)dB_s,$$

where $u, v \in \mathcal{V}$.

Theorem

Let X_t be an Itô process given by

$$X_t = X_0 + \int_0^t u(s,\omega)ds + \int_0^t v(s,\omega)dB_s.$$

Let $g(t,x) \in C^2([0,\infty) \times \mathbb{R})$. Then

$$Y_t = g(t, X_t)$$

is again an Itô process, and

$$\begin{aligned} Y_t = & Y_0 + \int_0^t g_1(s,X_s) + g_2(s,X_s) \cdot u(s,\omega) + \frac{1}{2} g_{22}(s,X_s) \cdot v^2(s,\omega)ds \\ & + \int_0^t g_2(s,X_s) \cdot v(s,\omega)dB_s, \end{aligned}$$

where $g_1 := \frac{\partial g}{\partial t}$, $g_2 := \frac{\partial g}{\partial x}$, $g_{22} := \frac{\partial^2 g}{\partial x^2}$, $\left(g_{12} = g_{21} := \frac{\partial^2 g}{\partial t \partial x}\right.$ and $\left. g_{11} := \frac{\partial^2 g}{\partial t^2}\right)$.

Example

Let B_t be the 1-dimensional Brownian motion on $(\Omega, \mathcal{F}, P)$ and consider the Itô integral

$$I = \int_0^t B_s dB_s$$

from the introduction.

Example

Let B_t be the 1-dimensional Brownian motion on $(\Omega, \mathcal{F}, P)$ and consider

$$\int_0^t s dB_s.$$

Theorem

Suppose $f(s,\omega) = f(s)$ (e.g. f is deterministic) and $f \in \mathcal{C}^2[0,t]$. Then

$$\int_0^t f(s)dB_s = f(t)B_t - \int_0^t f'(s)B_s ds.$$

References

Øksendal, B. (2003). *Stochastic differential equations* (pp. 43-48). Springer Berlin Heidelberg.

Steele, J. M. (2001). *Stochastic calculus and financial applications* (Vol. 45). SpringerScience & Business Media.