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Some properties of the Itô integral

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Key words

Basic properties of the Itô integral

Brownian Motion - martingale

Itô integral - t continuous

Basic properties of the Itô integral

Let $f, g \in \nu(0, T)$ and let $0 \leq S < U < T$.Then

(i) $\int_S^T f \, dB_t = \int_S^U f \, dB_t + \int_U^T f \, dB_t$ for a.a. ω

(ii) $\int_S^T (cf + g) \, dB_t = c \cdot \int_S^T f \, dB_t + \int_S^T g \, dB_t$ (c constant)

(iii) $E[\int_S^T f \, dB_t] = 0$

(iv) $\int_S^T f \, dB_t$ is $\mathcal{F}_T$ measurable.

(i) $\int_S^T \phi_n \, dB_t = \sum_{S \leq t_j \leq T} e_j(\omega)[B_{t_{j+1}} - B_{t_j}]$

$$= \sum_{S \leq t_j \leq U} e_j(\omega)[B_{t_{j+1}} - B_{t_j}] + \sum_{U \leq t_j \leq T} e_j(\omega)[B_{t_{j+1}} - B_{t_j}]$$

(ii) clear because of the linearity of summation

(iv) e_j and $B_{t_{j+1}} - B_{t_j}$ are $\mathcal{F}_{t_j}$ measurable

(iii) $E[\int_S^T \phi_n \, dB_t] = E[\sum_{S \leq t_j \leq T} e_j(\omega)(B_{t_{j+1}} - B_{t_j})]$

$$= \sum_{S \leq t_j \leq T} E[e_j(\omega)(B_{t_{j+1}} - B_{t_j})]$$

$$= \sum_{S \leq t_j \leq T} E[e_j(\omega)] E[(B_{t_{j+1}} - B_{t_j})]$$

$$= 0$$

- $B_{t_{j+1}} - B_{t_j} \sim N(0, t_{j+1} - t_j)$

Definition: martingale

An n-dimensional stochastic process $\{M_t\}_{t\geq 0}$ on $(\Omega,\mathcal{F},P)$ is called a martingale with respect to a filtration $\{\mathcal{M}_t\}_{t\geq 0}$ if

(i) M_t is $\mathcal{M}_t$ -measurable for all t

(ii) $E[|M_t|] < \infty$ for all t

(iii) $E[M_s|\mathcal{M}_t] = M_t$ for all s$\geq$t

Brownian Motion - Martingale

(i) B_t is $\mathcal{F}_t$ -measurable for all t

(ii) $E[|B_t|]^2 < E[|B_t|^2] = t < \infty$ for all t

(iii) $E[B_s|\mathcal{F}_t] = E[B_s - B_t + B_t|\mathcal{F}_t]$
$= E[B_s - B_t|\mathcal{F}_t] + E[B_t|\mathcal{F}_t] = 0 + B_t = B_t$ for all $s \geq t$

- $E[aX + bY|\mathcal{H}] = aE[X|\mathcal{H}] + bE[Y|\mathcal{H}]$
- $E[X|\mathcal{H}] = E[X]$ if X is independent of $\mathcal{H}$
- $E[X|\mathcal{H}] = X$ if X is $\mathcal{H}$ -measurable

Doob's martingale inequality

If M_t is a martingale such that $t \to M_t(\omega)$ is continuous a.s.,then for all $p \geq 1$, $T \geq 0$ and all $\lambda > 0$

$$P(\sup_{0 \leq t \leq T} |M_t| \geq \lambda) \leq \frac{1}{\lambda^p} \cdot E[|M_T|^p] \tag{1}$$

Itô integral - t continuous

Let $f \in \nu(0, T)$. Then there exists a t-continuous version of $\int_0^t f(s,\omega)\, dB_s(\omega); 0 \leq t \leq T$, i.e. there exists a t-continuous stochastic process J_t on $(\Omega, \mathcal{F}, P)$ such that

$$P(J_t = \int_0^t f\, dB) = 1 \quad \forall t, 0 \leq t \leq T.$$

Proof. Let ϕ_n be the elementary function such that

$$E[\int_0^T (f - \phi_n)^2 \, dt] \to 0, when\ n \to \infty.$$

Put $I_n(t,\omega) = \int_0^t \phi_n(s,\omega) \, dB_s(\omega)$

$$I_t = I(t,\omega) = \int_0^t f(s,\omega) \, dB_s(\omega); 0 \leq t \leq T.$$

Then $I_n(\cdot,\omega)$ is continuous for all n.

$I_n(s,\omega)$ is a martingale with respect to $\mathcal{F}_t$ for all n ($s > t$):

$$E[I_n(s,\omega)|\mathcal{F}_t] = E[(\int_0^t \phi_n \, dB + \int_t^s \phi_n \, dB)|\mathcal{F}_t]$$

$$= \int_0^t \phi_n \, dB + E[\sum_{t \leq t_j \leq s} e_j \Delta B_j | \mathcal{F}_t]$$

$$= \int_0^t \phi_n \, dB + \sum_j E[E[e_j \Delta B_j | \mathcal{F}_{t_j}]|\mathcal{F}_t]$$

$$= \int_0^t \phi_n \, dB + \sum_j E[e_j E[\Delta B_j | \mathcal{F}_{t_j}]|\mathcal{F}_t]$$

$$= \int_0^t \phi_n \, dB$$

$$= I_n(t,\omega)$$

Hence $I_n - I_m$ is also an $\mathcal{F}_t$ martingale. So by Doob's martingale inequality we have

$$P(\sup_{0\leq t\leq T} |I_n(t,\omega) - I_m(t,\omega)| \geq \epsilon)$$

$$\leq \frac{1}{\epsilon^2} \cdot E[|I_n(T,\omega) - I_m(T,\omega)|^2]$$

$$= \frac{1}{\epsilon^2} \cdot E[\int_0^T (\phi_n - \phi_m)^2 \, ds] \to 0 \text{ as } m, n \to \infty$$

$$\Rightarrow P(\sup_{0\leq t\leq T} |I_{n_{k+1}}(t,\omega) - I_{n_k}(t,\omega)| > 2^{-k}) < 2^{-k}$$

for a subsequence $n_k \to \infty$.

By the Borel-Cantelli lemma
$P(\sup_{0\leq t\leq T} |I_{n_{k+1}} - I_{n_k}| > 2^{-k} \text{ for infinitely many k}) = 0$

So for almost all ω there exists $k_1(\omega)$ such that
$\sup_{0\leq t\leq T} |I_{n_{k+1}}(t,\omega) - I_{n_k}(t,\omega)| \leq 2^{-k}, k \geq k_1(\omega)$

Therefore $I_{n_k}(t,\omega)$ is uniformly convergent for $t \in [0, T]$ for almost all ω and the limit is t-continuous for $t \in [0, T]$.

$\square$

Corollary

Let $f(t, \omega) \in \nu(0, T)$ for all T. Then

$$M_t(\omega) = \int_0^t f(s, \omega)\, dB_s$$

is a martingale w.r.t. $\mathcal{F}_t$ and for $\lambda, T > 0$

$$P(\sup_{0 \leq t \leq T} |M_t| \geq \lambda) \leq \frac{1}{\lambda^2} \cdot E[\int_0^T f(s, \omega)^2\, ds].$$

Literature

Øksendal, *B.*(2010) Stochastic Differential Equations:
An Introduction with Applications. Springer