Dr. Tim Brereton
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Summer Term 2015

Stochastic Simulation

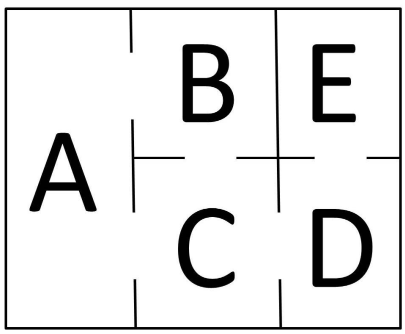
Problem Sheet 1

Deadline: April 23, 2015 at noon before the exercises

Please email your code to lisa.handl@uni-ulm.de AND hand in a printed copy of the code!

Exercise 1 (theory) (3 + 2 + 2 + 1 points)

An apartment has 5 rooms, A, B, C, D and E, which are connected as shown in the following plan (rooms are connected if there is a gap in the wall separating them).



A cat walks through these rooms at random, starting in room A. If the cat goes to another room, it uniformly selects one of the directly neighboring rooms. Let X_n be the room in which the cat is after switching rooms for the n-th time.

a) Draw a graph representing the Markov chain $\{X_n\}_{n\in\mathbb{N}}$ and write out its transition matrix.

b) Calculate the probability that the cat is in room C after switching twice, i.e., that $X_2 = C$.

c) Calculate the probability that $X_6 = B$ given that $X_4 = B$.

d) State if the Markov chain is irreducible and justify your answer.

Exercise 2 (theory) (3 points)

Let $\{X_n\}_{n\in\mathbb{N}}$ be a Markov chain with state space $E = \{1, 2, 3\}$ and transition matrix $P = (p_{i,j})$, where $p_{12} = p_{23} = p_{31} = 1$. The initial distribution is $\boldsymbol{\alpha} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Now define

$$Y_n = \begin{cases} 0, & \text{if } X_n = 1, \\ 1, & \text{otherwise.} \end{cases}$$

Show that the sequence $\{Y_n\}_{n\in\mathbb{N}}$ is not a Markov chain.

Exercise 3 (theory) (2 + 2 points)

Consider a Markov chain with state space $E = \{0, 1, 2\}$ and transition matrix

$$P = \begin{pmatrix} \frac{1}{8} & \frac{3}{8} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{1}{2} & \frac{1}{8} \end{pmatrix}.$$

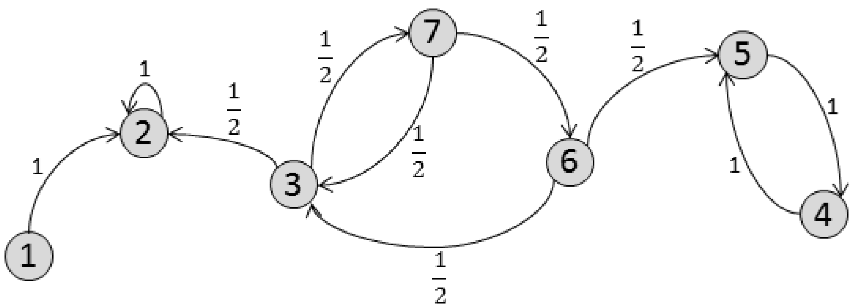
Find a random mapping representation of P using

a) $Z \sim \mathrm{U}(0, 1)$.

b) $Z \sim \mathrm{Bin}(3, \frac{1}{2})$, i.e., Z binomial with parameters $n = 3$ and $p = \frac{1}{2}$.

Exercise 4 (theory) (3 + 1 points)

Consider a Markov chain whose transition matrix, P, is defined by the following graph



a) Find the communicating classes of P. Which of these classes are closed?

b) State if the transition matrix is irreducible and justify your answer.

Exercise 5 (programming) (2 + 1 points)

Consider Example 2 from the lecture: A flea lives in a house with three dogs. Every day, it either stays where it is (with probability 0.5) or jumps (with probability 0.5) to one of the other dogs (selected uniformly).

a) Write a Matlab program to simulate this Markov chain using the initial distribution $\boldsymbol{\mu_0} = \boldsymbol{\delta_1}$. Run your simulation for $n = 365$ days and plot a histogram of the visited states.

   Hint: You can use the Matlab function `hist(x,y)` to draw a histogram of the entries in `x`, where `y` is a vector specifying the bin centers.

b) Calculate the distribution of the flea's position after 5, 10 and 365 days. Use the initial distribution from a).

Exercise 6 (programming) (3 points)

Consider a Markov chain with state space $E = \mathbb{N}$ and transition matrix $P = (p_{i,j})$, where

$$p_{i,j} = \begin{cases} \frac{i-1}{i}, & \text{if } j = i - 1, \\ \frac{1}{i}, & \text{if } j = i + 1, \\ 0, & \text{otherwise.} \end{cases}$$

Write a Matlab program to simulate this Markov chain using the initial distribution $\boldsymbol{\mu_0} = \boldsymbol{\delta_1}$. Generate and plot a realization of it up to $n = 30$.