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Summer Term 2015

Stochastic Simulation

Problem Sheet 2

Deadline: April 30, 2015 at noon before the exercises

Please email your code to lisa.handl@uni-ulm.de AND hand in a printed copy of the code!

Exercise 1 (theory) (3 + 3 points)

a) Which of the following random variables are stopping times in general, i.e., for any Markov chain $\{X_n\}_{n\in\mathbb{N}}$ with state space $\mathbb{Z}$?

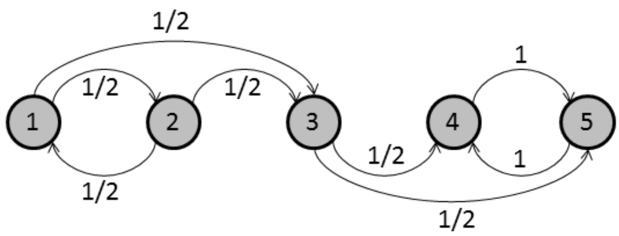
(i) $\tau_1 = \inf\{n \geq 0 : X_n + X_0 = 5\}$

(ii) $\tau_2 = 10$

(iii) $\tau_3 = \sup\{n \geq 0 : X_n \in \{1, 2, 3\}\}$

Justify your answer.

b) Consider the Markov chain $\{Y_n\}_{n\in\mathbb{N}}$ with initial distribution $\boldsymbol{\mu_0} = \boldsymbol{\delta_1}$ and the following transition graph.



Which of the following random variables are stopping times for $\{Y_n\}_{n\in\mathbb{N}}$?

(i) $\tau_4 = \inf\{n \geq 0 : Y_n + Y_{n+1} = 5\}$

(ii) $\tau_5 = \inf\{n \geq 0 : Y_{\lceil n/2 \rceil} \geq 4\}$

(iii) $\tau_6 = \sup\{n \geq 0 : Y_n \in \{1, 2, 3\}\}$

Justify your answer.

Exercise 2 (theory) (3 points)

Prove the following statement: If P is a stochastic matrix, then P^k is a stochastic matrix for every $k \in \mathbb{N}$.

Exercise 3 (theory) (2 + 2 points)

Sisyphus is trying to roll a big round rock up an infinitely high hill. In every step he gets one meter higher with probability $p \in (0,1)$ and he loses the rock (so that it rolls back to the bottom) with probability $1-p$. Of course, he starts at the bottom of the hill. Consider the Markov chain $\{X_n\}_{n\in\mathbb{N}}$ which is given by the height of Sisyphus and his rock.

a) Show that $\{X_n\}_{n\in\mathbb{N}}$ is recurrent.

b) Show that $\{X_n\}_{n\in\mathbb{N}}$ is positive recurrent.

Exercise 4 (theory) (3 points)

Consider a Markov chain $\{X_n\}_{n\in\mathbb{N}}$ with transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}.$$

Find the stationary distribution of $\{X_n\}_{n\in\mathbb{N}}$.

Exercise 5 (programming) (4 points)

Write a Matlab program to simulate $\{Y_n\}_{n\in\mathbb{N}}$ as defined in Exercise 1 b) and estimate the expectation of τ_4, τ_5 and τ_6 if they are stopping times of $\{Y_n\}_{n\in\mathbb{N}}$. Use a sample size of at least $N = 10^3$.

Exercise 6 (programming) (4 points)

Consider the Markov chain $\{X_n\}_{n\in\mathbb{N}}$ with state space $\mathbb{Z}$, initial distribution $\boldsymbol{\mu_0} = \boldsymbol{\delta_0}$ and transition probabilities

$$p_{i,j} = \begin{cases} \frac{1}{4}, & \text{if } j = i-1, \\ \frac{1}{2}, & \text{if } j = i, \\ \frac{1}{4}, & \text{if } j = i+1, \\ 0 & \text{otherwise.} \end{cases}$$

Simulate $\{X_n\}_{n\in\mathbb{N}}$ and estimate the probability that it hits 8 before it hits -1. Use a sample size of at least $N = 10^5$.