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Summer Term 2015

Stochastic Simulation

Problem Sheet 4

Deadline: May 21, 2015 at noon before the exercises

Please email your code to lisa.handl@uni-ulm.de AND hand in a printed copy of the code!

Exercise 1 (theory) (3 points)

Prove that the Metropolis-Hastings algorithm works, i.e., prove that (for any proposal matrix Q) the transition matrix of the Markov chain used in the Metropolis-Hastings algorithm is in detailed balance with $\boldsymbol{\pi}$.

Exercise 2 (theory) (4 points)

The acceptance-rejection algorithm introduced in the lecture works analogously for absolutely continous distributions. Suppose we have a density f we want to sample from and a density g we can easily sample from and there is a constant $C > 0$ such that $Cg(x) \geq f(x)\ \forall x \in \mathbb{R}$. Then we can draw from f as follows:

- Draw X according to g and $U \sim \mathcal{U}(0,1)$ (independently).
- Accept X if $U \leq \frac{f(X)}{Cg(X)}$, reject otherwise.

Show that this approach leads to the desired result, i.e., show that

$$P\left(X \leq x \;\middle|\; U \leq \frac{f(X)}{Cg(X)}\right) = F(x),$$

where F is the cumulative distribution function of f.

Hint: Use the continuous version of the law of total probability, i.e., if X is an absolutely continuous random variable with density f, then

$$P(A) = \int_{\mathbb{R}} P(A \mid X = x) f(x) \mathrm{d}x.$$

Exercise 3 (programming) (3 + 3 points)

Write a Matlab program to sample from the binomial distribution with parameters $n = 10$ and $p = 0.2$ using acceptance rejection and the following proposal distributions.

a) The uniform distribution on $\{0, \ldots, 10\}$.

b) The geometric distribution with parameter $q = 0.3$, i.e., $P(X = k) = (1 - q)^k q$.

   Hint: You can sample from $\text{Geo}(q)$ by drawing from an exponential distribution with parameter $\lambda = -\log(1 - q)$ and rounding off.

Draw at least $N = 10^4$ times from this distribution using a) and b) and plot histograms of the sampled values (relative frequencies), as well as of the desired binomial distribution.

Exercise 4 (programming) (3 + 4 + 2 points)

Suppose n employees who come to work by car every day share n parking lots at their company. Every morning when they come to work, they park their cars randomly (uniformly) on the n parking lots.

a) Write a Matlab program to sample from the distribution of the cars on the parking lots using acceptance-rejection and the uniform distribution on $\{1, \ldots, n\}^n$ as proposal distribution.

   Hint: You can use an array of length n to represent this, where the index is the number of the parking lot and the entry is the number of the employee occupying it.

b) Now assume that each employee, with probability $p = 0.1$, is ill and stays at home (they do this independently of one another). Adapt your algorithm to account for this using the uniform distribution on $\{0, \ldots, n\}^n$ for the proposals, where 0 indicates that a parking lot stays empty.

   Hint: Simulate which employees are ill first, then use acceptance-rejection.

c) Sample at least $N = 10^3$ times from the distributions in a) and b) and estimate the probability that the car of employee 1 stands next to the car of employee n, as well as the acceptance probability of your algorithm for $n = 3$, 5 and 10.

Exercise 5 (theory) (2 + 2 points)

a) Calculate the acceptance probability of the algorithm in Exercise 4 a) by hand (as a function of n). What happens as n tends to infinity?

b) Prove that if $U \sim \mathcal{U}(0, 1)$ and $n \in \mathbb{N}$, then $\lceil n \cdot U \rceil$ and $\lfloor n \cdot U \rfloor$ have (discrete) uniform distribution on $\{1, \ldots, n\}$ and $\{0, \ldots, n - 1\}$, respectively.

Exercise 6 (theory) (3 points)

Consider a Markov chain with state space $\mathcal{X}$ and transition matrix P, and two arbitrary probability measures, $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$, on $\mathcal{X}$. Show that

$$\|\boldsymbol{\mu} P - \boldsymbol{\nu} P\|_{TV} \leq \|\boldsymbol{\mu} - \boldsymbol{\nu}\|_{TV}$$

and explain why this means that a Markov chain can only get closer to its stationary distribution, assuming that one exists.

Exercise 7 (theory) (3 points)

Let $P \in \mathbb{R}^{n \times n}$ be a transition matrix which is irreducible and periodic. Show that $\frac{P+I}{2}$ (where I is the n-dimensional identity matrix) is still irreducible, has the same stationary distribution(s) as P, but is aperiodic.

Exercise 8 (theory) (3 + 2 points)

Let $X \sim \mathrm{N}(1,2)$, $Y \sim \mathrm{N}(2,8)$ and $Z \sim \mathcal{U}(0,1)$.

a) Provide at least three different examples of couplings of X and Y.

b) Is there a coupling of X and Z such that $\rho = \mathrm{corr}(X,Z) = 1$? Give an example or provide a proof that it is not possible.

Exercise 9 (theory) (5 points)

Consider a Markov chain with state space $\mathcal{X} = \{1,2\}$ and transition matrix

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix},$$

where $\alpha, \beta \in (0,1)$. Suppose you run two Markov chains, $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$, with transition matrix P independently, where $X_0 = 1$ and $Y_0 = 2$. Calculate the distribution of

$$\tau = \inf\{n \geq 0 : X_n = Y_n\}.$$

Exercise 10 (programming) (4 + 3 points)

Suppose you want to draw uniformly from the space Ω of all 5×5 matrices with values in $\{0,1\}$ and no 1s directly next to each other in a row or column.

a) Propose a suitable Markov chain for doing this using Markov chain Monte Carlo (changing only one entry at a time) and show that it is irreducible on Ω. Specify the state space, proposal distributions and acceptance probabilities.

b) Implement an MCMC algorithm for this problem in Matlab using a).