Dr. Tim Brereton
Lisa Handl

Summer Term 2015

Stochastic Simulation

Problem Sheet 5

Deadline: May 28, 2015 at noon before the exercises

Please email your code to lisa.handl@uni-ulm.de AND hand in a printed copy of the code!

Exercise 1 (theory) (3 points)

Let X, Y and Z be three discrete random variables with values in E, F and G, respectively. Prove the following statement:

If there is a function $g\colon E \times F \to [0,1]$ such that $P(X = x \mid Y = y, Z = z) = g(x,y)$ for all $x \in E, y \in F, z \in G$, then $P(X = x \mid Y = y) = g(x,y)$ for all $x \in E, y \in F$, and X and Z are conditionally independent given Y.

Exercise 2 (theory) (3 points)

Consider a Markov random field $\boldsymbol{X}$ on a finite set S with respect to the neighborhood system defined by the graph $G = (S, E)$. Let $G_A = (A, E_A)$ with $A \subset S$ and $E_A = \{(s_1, s_2) \in E : s_1, s_2 \in A\}$ be a subgraph of G. Show that if E contains no edges between A and $S \setminus A$, then $\boldsymbol{X}_A = \{X_s, s \in A\}$ is a Markov random field on G_A.

Exercise 3 (theory) (3 + 2 points)

Consider a Markov random field $\boldsymbol{X}$ on a finite set S with respect to the neighborhood system defined by the graph $G = (S, E)$. Let $M \subset S$ be the set of all vertices without neighbors. Show that

a) The set $\boldsymbol{X}_M = \{X_s, s \in M\}$ consists of independent random variables.

b) $\boldsymbol{X}_M$ is independent of $\boldsymbol{X}_{S\setminus M}$.

Exercise 4 (theory) (3 points)

Consider a Markov random field $\boldsymbol{X}$ on a finite set S with respect to the neighborhood system defined by the graph $G = (S, E)$. Let $\tilde{G} = (V, \tilde{E})$ be another graph with the same vertices as G but different edges. Is $\boldsymbol{X}$ a Markov random field with respect to the neighborhood system defined by $\tilde{G}$? Give a counter example or provide a proof of your answer.

Exercise 5 (programming) (3 + 2 points)

Consider the set A of all 5×5 matrices with values in $\{0, 1\}$ and the subset $A \subset \Omega$ of all such matrices which have no 1s directly next to each other in a row or column, like in Exercise 10 on the last problem sheet. This time, we do not want to draw uniformly from Ω but according to the distribution

$$\boldsymbol{\pi}_T(\boldsymbol{x}) = \frac{1}{Z_T} \mathrm{e}^{-\frac{1}{T}\mathcal{E}(\boldsymbol{x})} \quad \forall \boldsymbol{x} \in A$$

with the energy function

$$\mathcal{E}(\boldsymbol{x}) = \begin{cases} -\sum_{s \in S} x_s \log(\lambda) & \text{if } \boldsymbol{x} \in \Omega, \\ \infty & \text{otherwise,} \end{cases}$$

where $T = 1$ and $\lambda \in (0, \infty)$. Here S denotes the set of all sites in the matrix $\boldsymbol{x}$.

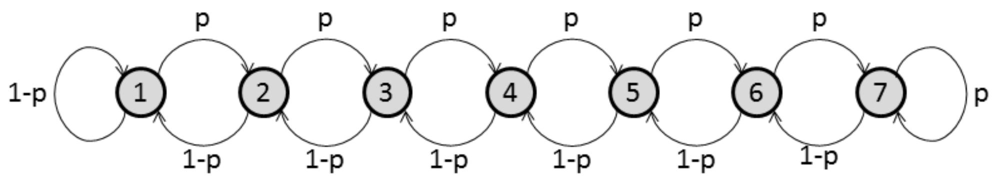
a) Write a Matlab program to draw from $\boldsymbol{\pi}$ approximately, using the Metropolis algorithm.

b) Run your program from a) for $\lambda = 0.5$, 1 and 1.5 and at least $N = 10^5$ steps and estimate the expected number of ones in the resulting random matrix.

Hint: You can do so by averaging over all matrices in the Markov chain you run.

Exercise 6 (programming) (2 + 3 + 2 points)

Consider two Markov chains $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ with the following transition graph.



a) Write out a random mapping representation of $\{X_n\}_{n\in\mathbb{N}}$ using Bernoulli distributed random variables.

b) Write a Matlab program to simulate the random mapping coupling of $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ for $p = 0.6$, $X_0 = 1$ and $Y_0 = 7$ using a) and plot a realization of it up to the coupling time $\tau_{\mathsf{couple}} = \inf\{n \geq 0 : X_n = Y_n\}$.

c) Run your program from b) at least $N = 10^4$ times and estimate the expected coupling time, $\mathbb{E}\tau_{\mathsf{couple}}$.