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Summer Term 2015

Stochastic Simulation

Problem Sheet 11

Deadline: July 16, 2015 at noon before the exercises

Please email your code to lisa.handl@uni-ulm.de AND hand in a printed copy of the code!

Definition (Compound Poisson Process)

A very useful extension of a Poisson process is what is called a *compound Poisson process*. A compound Poisson process replaces the unit jumps of a Poisson process with random jump sizes.

Given a homogeneous Poisson process $\{N_t\}_{t\geq 0}$ with intensity $\lambda \in (0, \infty)$ and a jump distribution G, we say that $\{X_t\}_{t\geq 0}$ with

$$X_t = \sum_{i=1}^{N_t} J_i \quad (X_t = 0 \text{ if } N_t = 0)$$

is a compound Poisson process, where the jumps $\{J_n\}_{n\in\mathbb{N}}$ are i.i.d. draws from G.

Exercise 1 (theory) (4 + 3 bonus points)

a) Show that a compound Poisson process has stationary and independent increments.

b) Let $\{N_t^{(1)}\}_{t\geq 0}$, $\{N_t^{(2)}\}_{t\geq 0}$ and $\{N_t^{(3)}\}_{t\geq 0}$ be three independent Poisson processes with intensity $\lambda \in (0, \infty)$ and define

$$X_t = N_t^{(1)} + N_t^{(2)} - N_t^{(3)} \quad \forall t \geq 0.$$

Express $\{X_t\}$ as a compound Poisson process, i.e., specify the underlying homogeneous Poisson process and its jump distribution.

Exercise 2 (theory) (2 + 2 bonus points)

Let $\{N_t\}_{t\geq 0}$ be an inhomogeneous Poisson process with intensity function

$$\lambda(t) = 2\exp\left(-\frac{x}{10}\right) \quad \forall x \geq 0.$$

a) Calculate the probability $P(N_{10} > 7 \mid N_5 = 7)$.

b) Find $P(N_{10} - N_5 = 1 \mid N_6 = 3)$.

Hint: Note that when X is a Poisson distributed random variable and there is a decomposition of X into a sum of independent random variables Y and Z, i.e., $X \overset{d}{=} Y + Z$, then Y and Z are Poisson distributed as well.

Exercise 3 (programming) (4 + 2 bonus points)

a) Write a Matlab program to simulate an inhomogeneous Poisson process $\{N_t\}_{t\geq 0}$ with rate function $\lambda(t) = 3(\cos(t) + 1)$ in the interval $[0, 10]$

   1. by thinning a homogeneous Poisson process
   2. by using a grid with a mesh size of $h = 0.01$

   and plot one of its realizations (for each method). Add the rate function to your plot.

b) Estimate the expected value of $N_{3\pi}$ using a sample size of at least $N = 10^4$. What is the true expected value?

Exercise 4 (programming) (3 + 1 bonus points)

Suppose you go fishing at some lake and the fish bite according to a Poisson process with rate $\lambda = 1.5$. The weight of the fish (in pounds) is distributed like $\frac{1}{4}Y$, where Y has a χ^2 distribution with 4 degrees of freedom. Since you're very talented you never lose a fish that bites.

a) Let X_t be the total weight of the fish you caught until time t (in hours). Write a Matlab program to simulate the process $\{X_t\}_{t\geq 0}$ and plot a realization of it up to 5 hours.

b) Estimate the probability that you catch more than 6 pounds of fish if you go fishing for 5 hours.