Random Fields I **Universität Ulm**
SoSe 2016 Dr. Patricia Alonso Ruiz
April 11, 2016 Dr. Vitalii Makogin

Exercise sheet 1 (total – 17 points) till April 19, 2016

Exercise 1-1 (4 points)

Prove the existence of a random field $X = \{X(t), t \in T\}$ with the following finite dimensional distributions and specify the measurable spaces $(E_{t_1,\dots,t_n}, \mathcal{E}_{t_1,\dots,t_n})$.

1. (2 points) $X(t) \sim Poi(\lambda_t), \lambda_t > 0, t \in T$ and $X(s), X(t)$ are independent for every $t, s \in T, s \neq t$.
2. (2 points) $X(t) = |Y(t)|^2$, where $Y(t) \sim N(0, I_n)$, I_n denotes the identity matrix and $Y(s), Y(t)$ are independent for every $t, s \in T, s \neq t$. Find the distribution of $X(t)$.

Hint: Sometimes it's easier to use Kolmogorov's existence theorem formulated in terms of characteristic functions. For example, one can use Proposition A. [1]

Exercise 1-2 (2 points)

Give an example (not the one presented in the lecture) of a non-continuous random function which has a continuous modification.

Exercise 1-3 (4 points)

Let $Y \sim U[0,1]$ and Z be a $d-$dimensional random vector, where Z and Y are independent. Consider a random field $X = \{X(t), t \in \mathbb{R}^d\}$ defined by $X(t) = \sqrt{2}\cos(2\pi Y + \langle t, Z\rangle)$, where $\langle \cdot, \cdot \rangle$ denotes the scalar product. Each realization of X is a cosine wave function.

1. (2 points) Compute the expectation of $X(t)$ for $t \in \mathbb{R}^d$.
2. (2 points) Determine the covariance function of X.

Exercise 1-4 (2 points)

Let Φ be a homogeneous Poisson point process in $\mathbb{R}^d$ with intensity $\lambda > 0$.

1. (1 point) Write down the finite dimensional distributions of Φ for disjoint bounded Borel sets $B_1, \dots, B_n$.
2. (1 point) Compute the expectation $\mathbf{E}\Phi(B)$ and the variance $\mathbf{Var}\Phi(B)$ for bounded Borel sets B.

[1]**Proposition A.** The family of measures $\mathbf{P}_{t_1,\dots,t_d}$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)), (t_1, \dots, t_d) \in T^d, d \geq 1,$ satisfies the conditions of symmetry and consistency iff for all $d \geq 2, (s_1, \dots, s_d) \in \mathbb{R}^d$ and $(t_1, \dots, t_d) \in T^d$ it holds

$$\varphi_{\mathbf{P}_{t_1,\dots,t_d}}((s_1, \dots, s_d)) = \varphi_{\mathbf{P}_{t_{i_1},\dots,t_{i_d}}}((s_{i_1}, \dots, s_{i_d}))$$

for any permutation $(1, \dots, d) \to (i_1, \dots, i_d)$,

$$\varphi_{\mathbf{P}_{t_1,\dots,t_{d-1}}}((s_1, \dots, s_{d-1})) = \varphi_{\mathbf{P}_{t_1,\dots,t_d}}((s_1, \dots, s_{d-1}, 0)).$$

Exercise 1-5 (5 points)

Let Φ be a homogeneous Poisson point process in $\mathbb{R}^d$ with intensity $\lambda > 0$. Consider the Shot-Noise-Field $\{X(t), t \in \mathbb{R}^d\}$ defined by

$$X(t) = \sum_{x \in \Phi} g(t - x),$$

where $g : \mathbb{R}^d \to \mathbb{R}$ is a deterministic function with $\int_{\mathbb{R}^d} |g(z)|dz < \infty$ and $\int_{\mathbb{R}^d} g^2(z)dz < \infty$.

1. (2 points) Prove that $\mathbf{E}X(t) = \lambda \int_{\mathbb{R}^d} g(z)dz, \forall t \in \mathbb{R}^d$.
2. (3 points) Prove that $\mathbf{Cov}[X(s)X(t)] = \lambda \int_{\mathbb{R}^d} g(t - s - z)g(-z)dz, \forall s, t \in \mathbb{R}^d$.

Hint: Firstly, prove the statements for a simple function g.