Random Fields I | **Universität Ulm**
SoSe 2016 | Dr. Patricia Alonso Ruiz
April 22, 2016 | Dr. Vitalii Makogin

Exercise sheet 2 (total – 20 points) till May 3, 2016

(revised)

Exercise 2-1 (4 points)

Prove that the following functions are positive definite

1. (2 points) Laplacian kernel: $C_1(x,y) = \exp\left(-\frac{1}{2}\sum_{i=1}^{d}|x_i - y_i|\right), x, y \in \mathbb{R}^d$.
   Hint: You may use Polya's theorem to prove that $e^{-|t|}$ is a positive definite function.
2. (2 points) Squared exponential c.f. $C_2(x,y) = \exp\left(-\frac{1}{2}\|x-y\|^2\right), x, y \in \mathbb{R}^d$.
   Hint: Use the Taylor's expansion of e^t and then prove that $e^{-<x,y>}$ is a positive definite function.

Exercise 2-2 (4 points)

1. (2 points) A real-valued random field $\{X(t), t \in \mathbb{R}^d_+\}$ is called H-self-similar ($H > 0$) if for any $a > 0$

   $$\{X(at), t \in \mathbb{R}^d_+\} \stackrel{d}{=} \{a^H X(t), t \in \mathbb{R}^d_+\},$$

   where $\stackrel{d}{=}$ denotes equality of finite-dimensional distributions.

   Prove that Lamperty transform of $H - ss$ random field X given by

   $$Y(t) = e^{-H<t,a>} X(e^{<t,a>}b), t \in \mathbb{R}^d, \text{ for } a, b \in \mathbb{R}^d_+ \setminus \{0\}$$

   is a strictly stationary random field.
2. (2 points) Consider a random field

   $$X(t) = Z_1 \cos(2\pi Y_1 + < t, Y_2 >) + Z_2 \sin(2\pi Y_1 + < t, Y_2 >), t \in \mathbb{R}^d,$$

   where Z_1, Z_2 are centered, uncorrelated, identically distributed random variables and Y_1, Y_2 are independent of Z_1, Z_2. Prove that X is a wide-stationary random field.

Exercise 2-3 (4 points)

Let $H \in (0,1)$ and $\{B^H(t), t \in \mathbb{R}^d_+\}$ be a real valued Gaussian H-self-similar random field with stationary increments.[1]

1. (2 points) Determine the distribution of $B^H(t)$.
2. (2 points) Compute the covariance function of B^H.

[1]**Def.** A random field X has stationary increments if for any $t, r \in \mathbb{R}^d_+$ $X(t+r) - X(t) \stackrel{d}{=} X(r)$.

Exercise 2-4 (3 points)

Recall that $f(x, y) = \frac{1}{2}\left(x^{2H} + y^{2H} - |x - y|^{2H}\right), x, y \in \mathbb{R}_+$ is the covariance function of a fractional Brownian motion $\{B^H(t), t \in \mathbb{R}_+\}$ with Hurst index $H \in (0, 1)$. Prove that the following functions are positive definite

1. (1 point) $C_3(x, y) = \frac{1}{2}\left(\|x\|^{2H} + \|y\|^{2H} - \|x - y\|^{2H}\right), x, y \in \mathbb{R}^d_+$.
2. (2 points) For every $\mathbf{H} = (H_1, \dots H_d) \in (0, 1)^d$

$$C_{\mathbf{H}}(t, s) = 2^{-d} \prod_{i=1}^{d} \left(t_i^{2H_i} + s_i^{2H_i} - |t_i - s_i|^{2H_i}\right), t = (t_1, \dots, t_d) \in \mathbb{R}^d_+, s = (s_1, \dots, s_d) \in \mathbb{R}^d_+.$$

   Write down $C_{\mathbf{H}}(t, s)$ for $\mathbf{H} = (0.5, \dots, 0.5)$.

Exercise 2-5 (5 points)

For a given vector $H = (H_1, \dots H_d) \in (0, 1)^d$ a fractional Brownian sheet $\{B_H(t), t \in \mathbb{R}^d_+\}$ with Hurst index H is a real-valued, centered Gaussian random field with covariance function given by

$$\mathbf{E}B_H(t)B_H(s) = 2^{-d} \prod_{i=1}^{d} \left(t_i^{2H_i} + s_i^{2H_i} - |t_i - s_i|^{2H_i}\right), t = (t_1, \dots, t_d) \in \mathbb{R}^d_+, s = (s_1, \dots, s_d) \in \mathbb{R}^d_+.$$

1. (1 point) Why does the fractional Brownian sheet exist?
2. (1 point) For every $t \in \mathbb{R}^d_+$ determine the distribution of $B_H(t)$.
3. (1 point) In the case $d = 2$ determine the distribution of the increment $B_H(t) - B_H(s)$.
4. (2 points) In the case $d = 2$ determine the distribution of a rectangular increment $\Delta_s B_H(t)$ given by

$$\Delta_s B_H(t) = (B_H(t_1, t_2) - B_H(s_1, t_2)) - (B_H(t_1, s_2) - B_H(s_1, s_2)).$$