Universität Ulm
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Exercise sheet 3 (total – 16 points) till May 17, 2016

Exercise 3-1 (4 points)

Let $X = \{X(t), t \in \mathbb{R}^d\}$ be a Gaussian random field which is continuous and differentiable in mean square sense. Let X'_h be its derivative.

1. (2 points) Prove that X'_h is again a Gaussian random field.
2. (2 points) Find the mean and covariance functions of X'_h.

Exercise 3-2 (3 points)

1. (2 points) Let $X = \{X(t), t \in \mathbb{R}^d\}$ be a wide sense stationary random field. Prove that its derivative X'_h is again wide sense stationary.
2. (1 point) Let $B^H = \{B^H(t), t \in \mathbb{R}^d_+\}$ be a fractional Brownian sheet (See Exercise 2-5). Prove that B^H is stochastically continuous.

Exercise 3-3 (2 points)

Prove that the function

$$f(x) = \left(\frac{1}{\|x\|_d + 1}\right)^\lambda, x \in \mathbb{R}^d$$

is positive semi-definite for any $\lambda > 0, d \in \mathbb{N}$.

Exercise 3-4 (2 points)

Using Bochner's theorem prove that the function $c(x) = (\|x\|_2^2 + a)^{-1}$, $x \in \mathbb{R}^2$ is positive semi-definite for any $a > 0$.

Exercise 3-5 (5 points)

Prove the following statement: A function $C : \mathbb{R} \to \mathbb{C}$ is real, continuous, and positive definite, if and only if it is the cosine transform of a measure F on $[0; \infty)$, i.e.

$$C(x) = \int_0^{+\infty} \cos(xt) dF(t) \text{ for all } x \in \mathbb{R}.$$

Compute the spectral density f of the following covariance functions $C : \mathbb{R} \to \mathbb{R}$:

1. $C(x) = \exp(-x^2)$,
2. $C(x) = \exp(-|x|)$,
3. $C(x) = (1 - |x|^2) I(-2 \leq x \leq 2)$.