Exercise sheet 4 (total – 12 points) till May 31, 2016

Exercise 4-1 (2 points)

For a random field $X = \{X(t), t \in T\}$ its expectation function $m : T \to \mathbb{R}$ is defined by $m(t) = \mathbf{E}X(t)$ for all $t \in T$ (if $\mathbf{E}X(t)$ exists). The (non-centered) covariance function $C : T \times T \to \mathbb{R}$ is defined by $C(s,t) = \mathbf{E}X(s)X(t)$ for all $s, t \in T$ (if it exists). Prove that a Gaussian random field is uniquely determined by its mean and covariance function.

Exercise 4-2 (2 points)

Let $X = \{X(t), t \in \mathbb{R}\}$ be a random polynomial with $X(t) = Y_0 + Y_1 t + \ldots + Y_n t^n$, $Y_i \sim N(0,1)$ i.i.d., $i = 0, \ldots, n$. Determine the expected value, the variance, the covariance function and the characteristic function of X.

Exercise 4-3 (2 points)

Let $X = \{X(t), t \in \mathbb{R}^d\}$ be a Shot-Noise Field (see Exercise 1-5), i.e.,

$$X(t) = \sum_{x \in \Phi} g(t - x).$$

Show, that X is wide-sense stationary.

Exercise 4-4 (2 points)

Prove that the function $C(x) = \cos(x), x \in \mathbb{R}$, is a valid covariance function.

Exercise 4-5 (4 points)

For any $a, b > 0$ and $m \in \mathbb{N}$, the so-called Cauchy family is given by

$$\varphi(h) = \frac{b}{(1 + (ah)^2)^m}, \quad h \in \mathbb{R}_+.$$

With the help of Theorem 2.1.17, prove that φ is positive semi definite and give the density of the corresponding finite measure on $\mathbb{R}_+$ (as long as it exists).

Hint: rewrite φ in integral form using the representation of the Gamma function $\Gamma(m) = 2\int_0^\infty e^{-r^2} r^{2m-1} dr$.