Random Fields I | **Universität Ulm**
SoSe 2016 | Dr. Patricia Alonso Ruiz
June 6, 2016 | Dr. Vitalii Makogin

Exercise sheet 5 (total – 15 points) till June 14, 2016

Exercise 5-1 (3 points)

Let B^H be a fractional Brownian sheet with index $(H_1, \ldots, H_d)$. Show that the following random field is a fractional Brownian sheet as well

$$B_2^H(t) = c_1^{H_1} \cdots c_d^{H_d} B^H\left(\frac{t_1}{c_1}, \ldots, \frac{t_d}{c_d}\right), (c_1, \ldots, c_d) \in (0, +\infty)^d.$$

Exercise 5-2 (3 points)

A random field $\{B_H(t), t \in \mathbb{R}_+^d\}$ is called a Lévy fractional Brownian field with Hurst index $H \in (0,1)$ if B_H is a centered Gaussian random field with covariance function

$$\mathbf{E} B_H(t) B_H(s) = \frac{1}{2}(\|t\|^{2H} + \|s\|^{2H} - \|t-s\|^{2H}).$$

Prove that $Y(t) = B_H(t + \mathbf{1}) - B_H(t), t \in \mathbb{R}_+^d$, is a stationary random field. Find its mean and covariance function.

Exercise 5-3 (3 points)

Let $\{B_H(t), t \in \mathbb{R}_+\}$ be a fractional Brownian motion with Hurst index $H \in (0,1)$. Let $C(k)$ be the covariance function of the process $Y(k) = B_H(k+1) - B_H(k), k \in \mathbb{N}$. Prove that

$$\sum_{k=1}^{\infty} |C(k)| \begin{cases} < \infty, & if\ H < 1/2 \\ = 0, & if\ H = 1/2 \\ = \infty & if\ H > 1/2. \end{cases}$$

Exercise 5-4 (3 points)

Show the existence of a stochastically continuous random field $X = \{X(t), t \in T\}$ which fulfils simultaneously the following conditions:

- The second moment does not exist,
- The variogram $\gamma(s,t)$ is finite for all $s, t \in T$.

Exercise 5-5 (3 points)

Let $X = \{X(t), t \in \mathbb{R}_+\}$ be a Lévy process (with stationary and independent increments). For some $T > 0$ take $K([0,T])$ to be the smallest ring that contains all finite unions of disjoint intervals in $[0,T]$. For $A = (s_1, t_1) \cup \ldots \cup (s_n, t_n)$, (s_i, t_i) pairwise disjoint, define

$$W(A) := \sum_{j=1}^{n} X(t_j) - X(s_j), \quad W(\{t\}) = 0 \quad \forall t \in [0,T].$$

Prove that W is an orthogonally scattered random measure on $([0,T], K([0,T]))$.