Random Fields I
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Exercise sheet 6 (total – 18 points) till June 28, 2016

Exercise 6-1 (3 points)

Let $X = \{X(t), t \in \mathbb{R}\}$ be a complex-valued process that is mean-square continuous and has independent increments, i.e., $\mathbf{E}(X(u) - X(s))\overline{(X(t) - X(u))} = 0$ for any $s < u < t$.

We introduce the family of random variables $W((a, b]) := X(b) - X(a)$ on the semiring $\mathcal{K} = \{(a, b], -\infty < a < b < \infty\}$. Show that W is an orthogonally scattered random measure on $\mathcal{K}$.

Exercise 6-2 (3 points)

Let $\mathbf{W}$ be a Gaussian white noise based on the Lebesgue measure and define a random field on $\mathbb{R}^d_+$ by setting $B(t) = \mathbf{W}([0, t])$, where $[0, t]$ is the rectangle $\prod_{i=1}^d [0, t_i]$.

1. Show that B is a centered Gaussian field on $\mathbb{R}^d_+$ with covariance

$$\mathbf{E}[B(s)B(t)] = \min(s_1, t_1) \cdots \min(s_d, t_d).$$

2. Suppose that $d > 1$ and fix $(d - k)$ of the index variables t_i. Show that B is a scaled $k-$parameter Brownian sheet in the remaining variables.

Exercise 6-3 (3 points)

Let $\{B(t), t \in \mathbb{R}^2_+\}$ be a Brownian sheet on the plane and denote by $[a, b]$ the rectangle $[a_1, b_1] \times [a_2, b_2]$. We introduce the family of random variables $W((a, b]) := B(b_1, b_2) - B(b_1, a_2) - B(a_1, b_2) + B(a_1, a_2)$ on the semiring $\mathcal{K} = \{(a, b], -\infty < a_i < b_i < \infty\}$. Show that W is an orthogonally scattered random measure on $\mathcal{K}$.

Exercise 6-4 (5 points)

Let $\{B(t), t \in \mathbb{R}_+\}$ be a Brownian motion. Define the family of random variables $W((a, b]) := B(b) - B(a)$ on the semiring $\mathcal{K} = \{(a, b], -\infty < a < b < \infty\}$.

1. (1 point) Show that W is an orthogonally scattered random measure on $\mathcal{K}$.
2. (2 points) Let $I(f)$ be the stochastic integral of $f \in L^2(\mathbb{R})$ with respect to W. Show that $I(f)$ is a Gaussian random variable. Find $\mathbf{E}I(f)$ and $\mathbf{E}[I(f)^2]$.
3. (2 points) Prove that $I(f)$ is a Gaussian random variable for any orthogonally scattered Gaussian random measure W.

Exercise 6-5 (4 points)

Let (E, Σ, ν) be a measurable space, and let ν be a $\sigma-$finite measure given on a semiring $\mathcal{K}$ of subsets of E. Show that simple function of the form

$$f : E \to \mathbb{C}, f = \sum_{i=1}^m c_i \mathbf{1}_{B_i},$$

where $c_i \in \mathbb{C}, B_i \in \mathcal{K}(E), i = 1 \dots m$, $\cup_{i=1}^m B_i = E$, $B_i \cap B_j = \emptyset$ for $i \neq j$, are dense in $L^2(E, \nu)$.