Content

1. Basic definitions
2. A general CLT
3. CLT for stationary Gaussian sequences

Definitions

1. $(\Omega, \Sigma, \mathbb{P})$ denotes a probability space and $(H, ||\cdot||)$ a separable Hilbert space
2. X denotes an isonormal process, that is $X : H \to L^2(\Omega)$ is centered, Gaussian and unitary
3. $D : \mathfrak{D}(D) \to L^2(\Omega; H)$ and $DF := \sum_{i=1}^{m} \frac{\partial f}{\partial x_i}(X(h_1), ..., X(h_m))h_i$ for all $F = f(X(h_1), ..., X(h_m))$. $\overline{D}$ denotes the closure of D.
4. $\Delta : \mathfrak{D}(\Delta) \to L^2(\Omega)$ and $\Delta F := -\sum_{n=1}^{\infty} nP_nF$, where P_n is the orthogonal projection on the n-th Wiener Chaos.
5. $\Delta^{-1} : L^2(\Omega) \to \mathfrak{D}(\Delta)$, such that $\Delta\Delta^{-1}F = F - \mathbb{E}F$.
6. $\mathrm{Div} : \mathfrak{D}(\mathrm{Div}) \to L^2(\Omega)$, where $\mathrm{Div}(F)$ is the uniquely determined element fulfilling $\langle F, \mathrm{Div}(u)\rangle_{L^2(\Omega)} = \langle DF, u\rangle_{L^2(\Omega;H)}$ for all $F \in \mathfrak{D}(D)$

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Lemma

1. Let $F \in L^2(\Omega)$ then we have $F \in \mathfrak{D}(\Delta)$ if and only if $F \in \mathfrak{D}(\overline{D})$ and $DF \in \mathfrak{D}(\mathrm{Div})$. And in this case $\mathrm{Div}(DF) = -\Delta F$.
2. Let $F := (F_1, ..., F_m) \in \mathfrak{D}(\overline{D})^m$ and $\varphi : \mathbb{R}^m \to \mathbb{R}$ be continuously differentiable with bounded first derivate. Then we have $\varphi \circ F \in \mathfrak{D}(\overline{D})$ and $\overline{D}\varphi \circ F = \sum_{i=1}^{m} \frac{\partial \varphi}{\partial x_i}(F)\overline{D}F_i$

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Lemma

Let $F, G \in \mathfrak{D}(\overline{D})$.
Let $\varphi : \mathbb{R} \to \mathbb{R}$ be C^1 with bounded first derivatives.
Then we have

$$\mathbb{E}(F\varphi \circ G) = \mathbb{E}F\mathbb{E}\varphi \circ G + \mathbb{E}\varphi' \circ G\langle \overline{D}G, -\overline{D}\Delta^{-1}F\rangle_H.$$

Proof.

$$\begin{aligned}
\mathbb{E}\left((F-\mathbb{E}(F))(\varphi\circ G)\right) &= \mathbb{E}\left((\Delta\Delta^{-1}F)(\varphi\circ G)\right)\\
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Corollary

Let $F \in \mathfrak{D}(\overline{D})$ be centered. Then we have

$$\mathrm{Var}(F) = \mathbb{E}\langle \overline{D}F, -\overline{D}\Delta^{-1}F\rangle_H.$$

Proof. Let $\varphi(x) := x$. Then we have

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Proposition

Let $F \in \mathfrak{D}(\overline{D})$ be centered and assume that $\mathbb{E}F^2 = 1$.
Let $\varphi : \mathbb{R} \to \mathbb{R}$ be C^1 and assume that $|\varphi'| \leq K$ for a $K \in [0, \infty)$.
Then we have

$$|\mathbb{E}\varphi' \circ F - \mathbb{E}(F \cdot \varphi \circ F)| \leq K\mathbb{E}|1 - \langle \overline{D}F, -\overline{D}\Delta^{-1}F\rangle_H|.$$

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The assertions remains true if $\varphi \in \text{Lip}(K)$ and if one assumes in addition that F has a density, w.r.t. the one dimensional Lebesgue measure.

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&\leq K \cdot \mathbb{E}|1 - \langle \overline{D}F, -\overline{D}\Delta^{-1}F\rangle_H|.
\end{aligned}$$

Remark

The assertions remains true if $\varphi \in \mathrm{Lip}(K)$ and if one assumes in addition that F has a density, w.r.t. the one dimensional Lebesgue measure.

Proposition

Let $F \in \mathfrak{D}(\overline{D})$ be centered and assume that $\mathbb{E}F^2 = 1$.
Let $\varphi : \mathbb{R} \to \mathbb{R}$ be C^1 and assume that $|\varphi'| \leq K$ for a $K \in [0, \infty)$.
Then we have

$$|\mathbb{E}\varphi' \circ F - \mathbb{E}(F \cdot \varphi \circ F)| \leq K\mathbb{E}|1 - \langle \overline{D}F, -\overline{D}\Delta^{-1}F\rangle_H|.$$

Proof.

$$\begin{aligned}
|\mathbb{E}\varphi' \circ F - \mathbb{E}(F \cdot \varphi \circ F)| &= |\mathbb{E}\varphi' \circ F - \mathbb{E}F\mathbb{E}\varphi \circ F - \mathbb{E}\varphi' \circ F\langle \overline{D}F, -\overline{D}\Delta^{-1}F\rangle_H| \\
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Definition

The total variation distance d_{TV} is defined by

$$d_{\mathrm{TV}}(X;Y) := \sup_{B \in \mathfrak{B}(\mathbb{R})} |\mathbb{P}(X \in B) - \mathbb{P}(Y \in B)|,$$

for all random variables $X, Y : \Omega \to \mathbb{R}$.

Definition

Moreover, the Wasserstein distance d_{W} is defined by

$$d_{\mathrm{W}}(X;Y) := \sup_{\varphi \in \mathrm{Lip}(1)} |\mathbb{E}\varphi(X) - \mathbb{E}\varphi(Y)|,$$

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Theorem

Let $F \in \mathfrak{D}(\overline{D})$ be centered, introduce $\sigma^2 := \mathrm{Var}(F) \neq 0$ and $N \sim N(0, \sigma^2)$. Then we have

$$d_W(F; N) \leq \frac{\sqrt{2}}{\sigma\sqrt{\pi}} \mathbb{E}|\sigma^2 - \langle \overline{D}F, -\overline{D}\Delta^{-1}F \rangle_H|.$$

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We have $\sigma^2 = \mathbb{E}\langle \overline{D}F, -\overline{D}\Delta^{-1}F \rangle_H$.

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Proof

Let $M \sim N(0,1)$ then it has already been proven that

$$d_W(F;M) \le \sup\left(|\mathbb{E}\varphi'(F) - \mathbb{E}F\varphi(F)| : \varphi \in C^1(\mathbb{R}),\ |\varphi'| \le \sqrt{\frac{2}{\pi}} \right).$$

(The latter is true for any square integrable F, not just for our particular one.)

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d_W(F;N) &= \sigma \sup_{\varphi \in \mathrm{Lip}(1)} \left| \mathbb{E}\varphi\left(\frac{F}{\sigma}\right) - \mathbb{E}\varphi\left(\frac{N}{\sigma}\right) \right| \\
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&\leq \sigma\sqrt{\frac{2}{\pi}}\mathbb{E}\left|1 - \left\langle D\left(\frac{F}{\sigma}\right), -D\Delta^{-1}\left(\frac{F}{\sigma}\right)\right\rangle_H\right| \\
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Moreover, assume that F has a density.
Then we have

$$d_{\mathrm{TV}}(F; N) \leq \frac{2}{\sigma^2} \mathbb{E} |\sigma^2 - \langle \overline{D} F, -\overline{D} \Delta^{-1} F \rangle_H |.$$

This is proven absolutely analogously to the previous theorem.

Central Limit Theorem

Let $(F_m)_{m\in\mathbb{N}} \subseteq \mathfrak{D}(\overline{D})$ be a sequence of centered, non constant random variables, introduce $\sigma_m^2 := \mathrm{Var}(F_m)$ and let $\sigma^2 \in (0,\infty)$.
Moreover, assume that

$$\lim_{m\to\infty} \langle \overline{D}F_m, -\overline{D}\Delta^{-1}F_m\rangle_H = \sigma^2$$

in $L^1(\Omega)$.
Then we have

$$\lim_{m\to\infty} \frac{F_m}{\sigma_m} = N(0,1)$$

in law.

Proof

Firstly, we have

$$\lim_{m\to\infty} \sigma_m^2 = \lim_{m\to\infty} \mathrm{Var}(F_m) = \lim_{m\to\infty} \mathbb{E}\langle \overline{D}F_m, -\overline{D}\Delta^{-1}F_m\rangle_H = \sigma^2.$$

Now let N be standard Gaussian. Then we have

$$\begin{aligned}
\lim_{m\to\infty} d_W\left(\frac{F_m}{\sigma_m}; N\right) &\leq \lim_{m\to\infty} \frac{\sqrt{2}}{\sqrt{\pi}} \mathbb{E}\left|1 - \frac{1}{\sigma_m^2}\langle \overline{D}F_m, -\overline{D}\Delta^{-1}F_m\rangle_H\right| \\
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&\leq \lim_{m\to\infty} \frac{1}{\sigma_m^2}\frac{\sqrt{2}}{\sqrt{\pi}} |\sigma_m^2 - \sigma^2| + \frac{1}{\sigma_m^2}\frac{\sqrt{2}}{\sqrt{\pi}} \mathbb{E}\left|\sigma^2 - \langle \overline{D}F_m, -\overline{D}\Delta^{-1}F_m\rangle_H\right| \\
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Remark

1. We just proved a CLT without directly stating any independence assumptions.
2. How does one verify that $\lim_{m \to \infty} \langle \overline{D} F_m, -\overline{D} \Delta^{-1} F_m \rangle_H = \sigma^2$? How does one even choose the Hilbert space?

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Content

1. Basic Definitions
2. A general CLT
3. **CLT for stationary Gaussian sequences**

Definitions

1. γ denotes the standard Gaussian measure and W_n the hermite polynomials.
2. $f \in L^2(\mathbb{R}, \mathfrak{B}(\mathbb{R}), \gamma)$ is a fixed non-constant function such that $\int_{\mathbb{R}} f d\gamma = 0$.
3. $(a_n)_{n\in\mathbb{N}}$ is such that $f = \sum_{n=1}^{\infty} a_n W_n$, where the latter convergence is understood in $L^2(\mathbb{R}, \mathfrak{B}(\mathbb{R}), \gamma)$.
4. The Hermite rank of f is defined by $d_f := \inf\{n \in \mathbb{N} |\ a_1 = \ldots = a_{n-1} = 0,\ a_n \neq 0\} < \infty$
5. $(X_k)_{k\in\mathbb{Z}}$ denote a centered, stationary, Gaussian sequence, such that $\mathrm{Var}(X_k) = 1$ for one (and therefore every) $k \in \mathbb{N}$
6. $C(k) := \mathbb{E} X_k X_0$
7. $V_m := \frac{1}{\sqrt{m}} \sum_{k=1}^{m} f(X_k)$

Remark

1. $(f(X_k))_{k\in\mathbb{Z}}$ is also a stationary sequence. Particularly, it consists of identically random variables.
2. $\mathbb{E}f(X_k) = \int_{\mathbb{R}} f d\gamma = 0$.

Remark

One can show that there is a separable Hilbert space H, an isonormal Gaussian process X and a sequence $(e_k)_{k\in\mathbb{Z}}$ such that

1. The closure of the linear span of $(e_k)_{k\in\mathbb{Z}}$ is H.
2. $\langle e_k, e_j\rangle_H = C(k-j)$ for all $k, j \in \mathbb{Z}$.
3. $X_k = X(e_k)$ for all $k \in \mathbb{Z}$.

Then one proves, using Malliavin Calculus, the following CLT:

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Then one proves, using Malliavin Calculus, the following CLT:

CLT for Functionals of stationary Gaussian Sequences

Assume that $\sum_{k\in\mathbb{Z}} |C(k)|^{d_f} < \infty$. Then we have

$$\lim_{m\to\infty} V_m = N,$$

in law, where N is a centered Gaussian random variable, with

$$\mathrm{Var}(N) = \sum_{j=d_f}^{\infty} j! a_j^2 \sum_{k\in\mathbb{Z}} C(k)^j \in (0,\infty)$$

CLT for the Increments of Fractional Brownian Motion

Let $(B_t)_{t\in\mathbb{R}}$ be a fractional Brownian motion with Hurst index $H \in (0, 1 - \frac{1}{2q})$, where $q \in \mathbb{N}$, with $q \neq 1$. Moreover, introduce

$$X_k := B_k - B_{k-1}, \; k \in \mathbb{Z}.$$

and

$$V_m := \frac{1}{\sqrt{m}} \sum_{k=1}^{m} W_q(X_k).$$

Then we have

$$\lim_{m\to\infty} V_m = N,$$

in law, where N is a centered Gaussian random variable, with

$$\mathrm{Var}(N) = \frac{q!}{2^q} \sum_{k\in\mathbb{Z}} (|k+1|^{2H} + |k-1|^{2H} - 2|k|^{2H})^q.$$

Proof

1. By definition we have $\mathbb{E}B_t = 0$ and $\mathbb{E}(B_t B_s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$.
2. Introduce $C(\nu) := \mathbb{E}(X_k X_{k+\nu})$ for all $\nu \in \mathbb{Z}$ and an arbitrary $k \in \mathbb{Z}$.
3. A trivial calculation yields

$$C(\nu) = \frac{1}{2}(|\nu - 1|^{2H} + |\nu + 1|^{2H} - 2|\nu|^{2H})$$

Consequently, $(X_k)_{k\in\mathbb{Z}}$ is a centered, stationary and has unit variance.
4. We already know that $\int_{\mathbb{R}} W_q d\gamma = 0$.
5. Trivially $a_n = 0$ if $n \neq q$ and $a_n = 1$ if $n = q$.
6. Moreover, one verifies

$$C(\nu) = H(2H-1)|\nu|^{2H-2} + o(|\nu|^{2H-2}), \text{ as } \nu \to \infty$$

Finally, the latter implies that $\sum_{\nu\in\mathbb{Z}} |C(\nu)|^q < \infty$ which yields the claimed convergence result.
7. The expression for the variance is trivial as $a_n = 0$ if $n \neq q$ and $a_n = 1$ if $n = q$.

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