Stable Distributions
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Exercise sheet 1 (total – 18 points) till April 21, 2016

Exercise 1-1 (2 points)

Let X_1, X_2 be two i.i.d. r.v.'s with probability density φ. Find a probability density of $aX_1 + bX_2$, where $a, b \in \mathbb{R}$.

Exercise 1-2 (2 points)

Let X be a symmetric stable random variable and X_1, X_2 be its two independent copies. Prove that X is a strictly stable r.v., i.e., for any positive numbers A and B, there is a positive number C such that

$$AX_1 + BX_2 \stackrel{d}{=} CX.$$

Exercise 1-3 (3 points)

1. (1 point) Prove that $\varphi = \{e^{-|x|}, x \in \mathbb{R}\}$ is a characteristic function. (Check Pólya's criterion for characteristic functions.[1])
2. (2 points) Let X be a real r.v. with characteristic function φ. Is X a stable random variable? (Verify definition.)

Exercise 1-4 (5 points)

Let $f : \mathbb{R} \to \mathbb{R}_+$ be a function of the following form

$$f(x) = \begin{cases} \left(\frac{\sigma}{2\pi}\right)^{1/2} \frac{1}{(x-\mu)^{3/2}} \exp\left\{-\frac{\sigma}{2(x-\mu)}\right\}, x > \mu, \\ 0, x \leq \mu, \end{cases} \qquad \textbf{(Lévy distribution)}$$

where $\sigma > 0, \mu \in \mathbb{R}$.

1. (2 points) Let r.v. $X \stackrel{d}{=} \sigma Z^{-2} + \mu$, where $Z \sim N(0, 1)$. Show that X has the probability density f.
2. (3 points) Prove that X is a stable random variable.

Exercise 1-5 (6 points)

Let $g : \mathbb{R} \to \mathbb{R}_+$ be a function of the following form

$$g(x) = \frac{\sigma}{\pi((x-\mu)^2 + \sigma^2)}, x \in \mathbb{R} \qquad \textbf{(Cauchy distribution)}$$

where $\sigma > 0, \mu \in \mathbb{R}$.

1. (3 points) Let r.v. $Y \stackrel{d}{=} \sigma \frac{Z_1}{Z_2} + \mu$, where $Z_1, Z_2 \sim N(0, 1)$ are independent. Show that Y has the probability density g.
2. (3 points) Prove that Y is a stable random variable.

[1]**Pólya's theorem.** If φ is a real-valued, even, continuous function which satisfies the conditions $\varphi(0) = 1$, φ is convex for $t > 0$, $\lim_{t\to\infty} \varphi(t) = 0$, then φ is the characteristic function of an absolutely continuous symmetric distribution.