Stable Distributions | **Universität Ulm**
SoSe 2016 | Prof. Dr. Evgeny Spodarev
April 21, 2016 | Dr. Vitalii Makogin

Exercise sheet 2 (total — 17 points) till May 3, 2016

Exercise 2-1 (3 points)

Let real r.v. X be Lévy distributed (see Exercise Sheet 1, Ex. 1-4). Find the characteristic function of X. Give parameters $(\alpha, \sigma, \beta, \mu)$ for the stable random variable X.

Hint: You may use the following formulas. [1]

$$\int_0^\infty \frac{e^{-1/(2x)}}{x^{3/2}} \cos(yx)dx = \sqrt{2\pi} e^{-\sqrt{|y|}} \cos(\sqrt{|y|}), y \in \mathbb{R},$$
$$\int_0^\infty \frac{e^{-1/(2x)}}{x^{3/2}} \sin(yx)dx = \sqrt{2\pi} e^{-\sqrt{|y|}} \sin(\sqrt{|y|})\text{sign}y, y \in \mathbb{R}.$$

Exercise 2-2 (5 points)

Let Y be a Cauchy distributed r.v. (see Exercise Sheet 1, Ex. 1-5.)

1. (2 points) Show that $Y \stackrel{d}{=} \sigma \tan(U) + \mu$ where $\sigma > 0, \mu \in \mathbb{R}$, and U is a uniformly distributed on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
2. (4 points) Find the characteristic function of Y. Give parameters $(\alpha, \sigma, \beta, \mu)$ for the stable random variable Y.
   Hint: Use Cauchy's residue theorem.

Exercise 2-3 (2 points)

Let $X \sim S_1(\sigma, \beta, \mu)$ and $a > 0$. Is aX stable? If so, define new $(\alpha_2, \sigma_2, \beta_2, \mu_2)$ of aX.

Exercise 2-4 (3 points)

Let $X \sim N(0, \sigma^2)$ and A be a positive $\alpha-$stable r.v. Is the new r.v. AX stable, strictly stable? If so, find its stability index α_2.

Exercise 2-5 (4 points)

Let L be a positive slowly varying function, i.e., $\forall x > 0$

$$\lim_{t \to +\infty} \frac{L(tx)}{L(t)} = 1. \tag{1}$$

1. (2 points) Prove that $x^{-\varepsilon} \leq L(x) \leq x^{\varepsilon}$ for any fixed $\varepsilon > 0$ and all x sufficiently large.
2. (2 points) Prove that limit (1) is uniform in finite intervals $0 < a < x < b$.

Hint: Use a representation theorem:[2]
A function Z varies slowly iff it is of the form $Z(x) = a(x) \exp\left(\int_1^x \frac{\varepsilon(y)}{y} dy\right)$, where $\varepsilon(x) \to 0$ and $a(x) \to c < \infty$ as $x \to \infty$.

[1] Oberhettinger, F. (1973). Fourier transforms of distributions and their inverses: a collection of tables. Academic press, p.25

[2] Feller, W. (1973). An Introduction to Probability Theory and its Applications. Vol 2, p.282