Stable Distributions | **Universität Ulm**
SoSe 2016 | Prof. Dr. Evgeny Spodarev
May 12, 2016 | Dr. Vitalii Makogin

Exercise sheet 3 (total — 17 points) till May 19, 2016

(revised)

Definition 1 (Infinitely divisible distributions)**.** *A distribution function F is called infinitely divisible if for all $n \geq 1$, there is a distribution function F_n such that*

$$Z \stackrel{d}{=} X_{n,1} + \cdots + X_{n,n},$$

where $Z \sim F$ and $X_{n,k}, 1 \leq k \leq n$ are i.i.d. r.v.'s with the distribution function F_n.

Exercise 3-1 (3 points)

For the following distribution functions check whether they are infinitely divisible.

1. (1 point) Gaussian distribution.
2. (1 point) Poisson distribution.
3. (1 point) Gamma distribution.

Exercise 3-2 (3 points)

Find parameters (a, b, H) in the canonic Lévy-Khintchin representation of a characteristic function for

1. (1 point) Gaussian distribution.
2. (1 point) Poisson distribution.
3. (1 point) Lévy distribution.

Exercise 3-3 (2 points)

What is wrong with the following argument? If $X_1, \ldots, X_n \sim Gamma(\alpha, \beta)$ are independent, then $X_1 + \cdots + X_n \sim Gamma(n\alpha, \beta)$, so gamma distributions must be stable distributions.

Exercise 3-4 (4 points)

Let $X \sim S_\alpha(\lambda, \beta, \gamma)$. The modified parameters $(\lambda_M, \beta_M, \gamma_M)$ and $(\lambda_B, \beta_B, \gamma_B)$ are defined in Remark 2.2 in the lecture notes.

1. (2 points) Show that $\eta_M(s) = \log \mathbf{E} e^{isX}$ is continuous as a function of its parameters $(\lambda_M, \beta_M, \gamma_M)$.
2. (2 points) Find the limit of $\eta_B(s)$ as $\alpha \to 1 \pm 0$.

Exercise 3-5 (5 points)

Let $X_i, i \in \mathbb{N}$ be i.i.d. r.v.'s with a density symmetric about 0 and continuous and positive at 0. Prove

$$\frac{1}{n}\left(\frac{1}{X_1} + \cdots + \frac{1}{X_n}\right) \stackrel{d}{\to} X, n \to \infty,$$

where X is a Cauchy distributed random variable.

Hint: At first, apply Khintchin's theorem (T.2.2 in the lecture notes). Then find parameters a, b and a spectral function H from Gnedenko's theorem (T.2.3 in the lecture notes).