Stable Distributions
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Exercise sheet 4 (total — 14 points) till June 2, 2016

Exercise 4-1 (2 points)

Show that the sum of two independent stable random variables with different α-s is not stable.

Exercise 4-2 (2 points)

Let $X \sim S_\alpha(\lambda, \beta, \gamma)$. Using the weak law of large numbers prove that when $\alpha \in (1, 2]$, the shift parameter $\mu = \lambda\gamma$ equals $\mathbf{E}X$.

Exercise 4-3 (2 points)

Let X be a standard Lévy distributed random variable. Compute its Laplace transform

$$\mathbf{E}\exp(-\gamma X), \gamma > 0.$$

Exercise 4-4 (3 points)

Let $X \sim S_{\alpha'}(\lambda', 1, 0)$, and $A \sim S_{\alpha/\alpha'}(\lambda_A, 1, 0)$, $0 < \alpha < \alpha' < 1$ be independent. The value of λ_A is chosen s.t. the Laplace transform of A is given by $\mathbf{E}\exp(-\gamma A) = \exp(-\gamma^{\alpha/\alpha'}), \gamma > 0$. Show that $Z = A^{1/\alpha'}X$ has a $S_\alpha(\lambda, 1, 0)$ distribution for some $\lambda > 0$.

Exercise 4-5 (5 points)

Let $X \sim S_\alpha(\lambda, 1, 0), \alpha < 1$ and the Laplace transform of X be given by $\mathbf{E}\exp(-\gamma X) = \exp(-c_\alpha \gamma^\alpha), \gamma > 0$,where $c_\alpha = \lambda^\alpha / \cos(\pi\alpha/2)$.

1. (3 points) Show that
$$\lim_{x\to\infty} x^\alpha \mathbf{P}\{X > x\} = C_\alpha,$$
where C_α is a positive constant.
Hint: Use the Tauberian theorem.[1]

2. (2 points) Prove that
$$\mathbf{E}|X|^p < \infty, \text{ for any } 0 < p < \alpha,$$
$$\mathbf{E}|X|^p = \infty, \text{ for any } p \geq \alpha.$$

[1](Feller 1971 Theorem XIII.5.4.) If L is slowly varying at infinity and $\rho \in \mathbb{R}_+$, the following relations are equivalent

$$U(t) \sim \frac{1}{\Gamma(\rho+1)} t^\rho L(t), t \to \infty, \qquad \int_0^\infty e^{-\tau x} dU(x) \sim \frac{1}{\tau^\rho} L\left(\frac{1}{\tau}\right), \tau \to 0.$$