Stable Distributions | **Universität Ulm**
SoSe 2016 | Prof. Dr. Evgeny Spodarev
June 9, 2016 | Dr. Vitalii Makogin

Exercise sheet 5 (total — 14 points) till June 16, 2016

Exercise 5-1 (1 point)

Let X_1, X_2 be two independent α-stable random variables with parameters (λ, β, γ). Prove that $X_1 - X_2$ is a stable random variable and find its parameters $(\alpha_1, \lambda_1, \beta_1, \gamma_1)$.

Exercise 5-2 (3 points)

Let $X_1, \dots, X_n$ be i.i.d $S_\alpha(\lambda, \beta, \gamma)$ distributed random variables and $S_n = X_1 + \dots + X_n$. Prove that the limiting distribution of

1. (1 point) $n^{-1/\alpha} S_n, n \to \infty$, if $\alpha \in (0, 1)$;
2. (1 point) $n^{-1}(S_n - 2\pi^{-1}\lambda\beta n \log n) - \lambda\gamma, n \to \infty$, if $\alpha = 1$;
3. (1 point) $n^{-1/\alpha}(S_n - n\lambda\gamma), n \to \infty$, if $\alpha \in (1, 2]$;

is $S_\alpha(\lambda, \beta, 0)$.

Exercise 5-3 (4 points)

Let $X_1, X_2 \dots,$ be a sequence of i.i.d. random variables and let $p > 0$. Applying the Borel-Cantelli lemmas, show that

1. (2 points) $\mathbf{E}|X_1|^p < \infty$ if and only if $\lim_{n\to\infty} n^{-1/p} X_n = 0$ a.s.,
2. (2 points) $\mathbf{E}|X_1|^p = \infty$ if and only if $\limsup_{n\to\infty} n^{-1/p} X_n = \infty$ a.s.

Exercise 5-4 (3 points)

Let ξ be a non-negative random variable with the Laplace transform $\mathbf{E}\exp(-\lambda\xi) = \exp(-\lambda^\alpha), \lambda \geq 0$. Prove that

$$\mathbf{E}\xi^{\alpha s} = \frac{\Gamma(1-s)}{\Gamma(1-\alpha s)}, s \in (0, 1).$$

Exercise 5-5 (3 points)

Denote by

$$\tilde{f}(s) := \int_0^\infty e^{-sx} f(x) dx,$$

the Laplace transform of a real function f defined for all $s > 0$, whenever $\tilde{f}$ is finite. For the following functions find the Laplace transforms (in terms of $\tilde{f}$):
1. For $a \in \mathbb{R}$ $f_1(x) := f(x - a), x \in \mathbb{R}_+$, and $f(x) = 0, x < 0$.
2. For $b > 0$ $f_2(x) := f(bx), x \in \mathbb{R}_+$.
3. $f_3(x) := f'(x), x \in \mathbb{R}_+$.
4. $f_4(x) := \int_0^x f(u) du, x \in \mathbb{R}_+$.