Stable Distributions | **Universität Ulm**
SoSe 2016 | Prof. Dr. Evgeny Spodarev
June 17, 2016 | Dr. Vitalii Makogin

Exercise sheet 6 (total — 15 points) till June 30, 2016

Exercise 6-1 (3 points)

Let $\tilde{f}, \tilde{g}$ be Laplace transforms of functions $f, g : \mathbb{R}_+ \to \mathbb{R}_+$.

1. (1 point) Find the Laplace transform of the convolution $f * g$.
2. (2 points) Prove the final value theorem: $\lim_{s\to 0} s\tilde{f}(s) = \lim_{t\to\infty} f(t)$.

Exercise 6-2 (2 points)

Let $\{X_n\}_{n\geq 0}$ be i.i.d. r.v.'s with a density symmetric about 0 and continuous and positive at 0. Applying the Theorem 2.8 from the lecture notes, prove that cumulative distribution function $F(x) := \mathbf{P}(X_1^{-1} \leq x), x \in \mathbb{R}$ belongs to the domain of attraction of a stable law G. Find its parameters $(\alpha, \lambda, \beta, \gamma)$ and sequences a_n, b_n s.t. $\frac{1}{b_n}\sum_{i=1}^n X_i^{-1} - a_n \xrightarrow{d} Y \sim G$ as $n \to \infty$.

Exercise 6-3 (3 points)

Let $\{X_n\}_{n\geq 0}$ be i.i.d. r.v.'s with for $x > 1$

$$\mathbf{P}(X_1 > x) = \theta x^{-\delta}, \quad \mathbf{P}(X_1 < -x) = (1-\theta)x^{-\delta},$$

where $0 < \delta < 2$. Applying the Theorem 2.8 from the lecture notes, prove that c.d.f. $F(x) := \mathbf{P}(X_1 \leq x), x \in \mathbb{R}$ belongs to the domain of attraction of a stable law G. Find its parameters $(\alpha, \lambda, \beta, \gamma)$ and sequences a_n, b_n s.t. $\frac{1}{b_n}\sum_{i=1}^n X_i - a_n \xrightarrow{d} Y \sim G$ as $n \to \infty$.

Exercise 6-4 (5 points)

Let X be a random variable with probability density function $f(x)$. Assume that $f(0) \neq 0$ and that $f(x)$ is continuous at $x = 0$. Prove that

1. (2 points) if $0 < r \leq \frac{1}{2}$, then $|X|^{-r}$ belongs to the domain of attraction of a Gaussian law,
2. (3 points) if $r > 1/2$ then $|X|^{-r}$ belongs to the domain of attraction of a stable law with stability index $1/r$.

Exercise 6-5 (2 points)

Find a distribution F which has infinite second moment and yet it is in the domain of attraction of the Gaussian law.