Stable Distributions | **Universität Ulm**
SoSe 2016 | Prof. Dr. Evgeny Spodarev
July 1, 2016 | Dr. Vitalii Makogin

Exercise sheet 7 (total — 14 points) till July 14, 2016

Exercise 7-1 (3 points)

Prove the following statement which is used in the proof of Proposition 2.3 in the Lecture notes.

Let $X \sim S_\alpha(\lambda, \beta, 0)$ with $\alpha \in (0, 2)$. Then there exist two i.i.d. r.v.'s Y_1 and Y_2 with common distribution $S_\alpha(\lambda, 1, 0)$ s.t.

$$X \stackrel{d}{=} \begin{cases} \left(\frac{1+\beta}{2}\right)^{1/\alpha} Y_1 - \left(\frac{1-\beta}{2}\right)^{1/\alpha} Y_2, & \text{if } \alpha \neq 1, \\ \left(\frac{1+\beta}{2}\right) Y_1 - \left(\frac{1-\beta}{2}\right) Y_2 + \frac{\lambda}{\pi}\left((1+\beta)\log\frac{1+\beta}{2} - (1-\beta)\log\frac{1-\beta}{2}\right), & \text{if } \alpha = 1. \end{cases}$$

Exercise 7-2 (3 points)

Prove that for $\alpha \in (0, 1)$ and fixed λ, the family of distributions $S_\alpha(\lambda, \beta, 0)$ is stochastically ordered in β, i.e., if $X_\beta \sim S_\alpha(\lambda, \beta, 0)$ and $\beta_1 \leq \beta_2$ then $\mathbf{P}(X_{\beta_1} \geq x) \leq \mathbf{P}(X_{\beta_2} \geq x)$ for $x \in \mathbb{R}$.

Exercise 7-3 (3 points)

Prove Exercise 2.9 in the Lecture Notes: Show that if $\frac{n}{b_n^2}\mu(b_n x) \sim Cx^{-\alpha}, n \to \infty$, where $\mu(x) = \int_{-x}^{x} y^2 dF(y)$, then

$$\begin{cases} n(F(b_n x) - 1) \to c_1 x^{-\alpha}, \\ nF(-b_n x) \to c_2 x^{-\alpha}, \end{cases} \quad \text{as } n \to \infty.$$

Exercise 7-4 (5 points)

Prove the following theorem.

Theorem 1. *A distribution function F is in the domain of attraction of a stable law with exponent $\alpha \in (0, 2)$ if and only if there are constants $C_+, C_- \geq 0, C_+ + C_- > 0$, such that*

1.
$$\lim_{y\to+\infty} \frac{F(-y)}{1 - F(y)} = \begin{cases} C_-/C_+, & \text{if } C_+ > 0, \\ +\infty, & \text{if } C_+ = 0, \end{cases}$$

2. *and for every $a > 0$*

$$\begin{cases} \lim_{y\to+\infty} \frac{1-F(ay)}{1-F(y)} = a^{-\alpha}, & \text{if } C_+ > 0, \\ \lim_{y\to+\infty} \frac{F(-ay)}{F(-y)} = a^{-\alpha}, & \text{if } C_- > 0. \end{cases}$$