1. Exercise sheet
Deadline: April, 26th, 16:15

Exercise 1: Examples of random closed sets
(2+1+2+2+1=8 Credits)

a) Let $X_1, \ldots, X_n$ be E-valued random variables. Show that $\{X_1, \ldots, X_n\}$ is a random closed set.

b) Let $X_1, \ldots, X_n$ be i.i.d. E-valued random variables. Let P denote their distribution, i.e. the induced probability measure $\mathbb{P}_{X_1}$ on E. Show that the capacity functional of $Z = \{X_1, \ldots, X_n\}$ is given by

$$T_Z(C) = 1 - (1 - P(C))^n, \quad C \in \mathcal{C}.$$

c) The *element-wise sum* or *Minkowski sum* of two set $A, B \subseteq \mathbb{R}^d$ is defined by

$$A + B := \{a + b \mid a \in A,\, b \in B\}.$$

Show that the Minkowski sum $C + F$ of a compact set $C \subseteq \mathbb{R}^d$ and a closed set $F \subseteq \mathbb{R}^d$ is closed.

d) Let V be a random vector in $\mathbb{R}^d$ and let $F \subseteq \mathbb{R}^d$ be a deterministic closed set. Show that F shifted by V, i.e. $F + \{V\}$ is a random closed set.

e) Let P be the distribution of V. Show that the capacity functional of $Z := F + \{V\}$ is given by

$$T_Z(C) = P(C + F^*), \quad C \in \mathcal{C},$$

where $F^* := \{-x \mid x \in F\}$ is the reflection of F at the origin.

Exercise 2: The Matheron-σ-algebra
(4 Credits)
Show that the σ-algebra generated by

$$\mathcal{F}_G, \quad G \in \mathcal{G},$$

is contained in the Matheron-σ-algebra. Where is the essential difficulty in showing that both σ-algebras are equal?
You may use without proof the following topological facts valid in seperable, locally compact metric spaces E:

- *For every open set $G \in \mathcal{G}$ there is a sequence $(C_i)_{i\in\mathbb{N}}$ of compact sets such that $C_i \subseteq C_{i+1}$ for all $i \in \mathbb{N}$ and $\bigcup_{i=1}^{\infty} C_i = G$.*
- *For every compact set $C \in \mathcal{C}$ there is a sequence $(G_i)_{i\in\mathbb{N}}$ of open sets such that $G_i \supseteq G_{i+1}$ for all $i \in \mathbb{N}$ and $\bigcap_{i=1}^{\infty} G_i = C$.*

Exercise 3: Construction of a semimetric of closed convergence
(2+2+2=6 Credits)
While we know by the Urysohn theorem that a metric of closed convergence exists, we want now to construct a corresponding semimetric explicitly. A *semimetric* on a set M is a function $d : M \times M \to \mathbb{R}$ which is

$$\text{positive definite: } \begin{cases} d(x,y) \geq 0 & x,y \in M \\ d(x,y) = 0 \iff x = y & x,y \in M \end{cases}$$
$$\text{and symmetric: } d(x,y) = d(y,x),$$

but which does not necessarily fulfill the triangular inequality.

Choose a sequence $(C_n)_{n\in\mathbb{N}}$ of compact sets in E with $C_n \subseteq \operatorname{int} C_{n+1}$ for all $n \in \mathbb{R}^d$ and $\bigcup_{n=1}^{\infty} C_n = E$, where $\operatorname{int} C$ denotes the interior of C. In $E = \mathbb{R}^d$ such a sequence is given by $C_n = [-n, n]^d$. It is not trivial that such a sequence exists in any locally compact seperable metric space; however it is true and should be used without proof here.

The *diameter* of a set $A \subseteq E$ is defined by

$$\operatorname{diam} A := \sup\{d(x, y) \mid x, y \in A\}.$$

If A is compact, the supremum is a maximum and hence it is finite. Choose a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $(0, \infty)$ with

$$\sum_{n=1}^{\infty} \alpha_n \cdot \operatorname{diam} C_n < \infty.$$

For $A \subseteq E$ and $\epsilon \in (0, \infty)$ we put

$$A_{\oplus\epsilon} := \{y \in E \mid \text{ there is } x \in A \text{ with } d(x, y) \leq \epsilon\}.$$

For $A, B \in \mathcal{F}$ put

$$\begin{aligned}
\tilde{d}_n(A, B) &:= \inf\{\epsilon > 0 \mid A \cap C_n \subseteq B_{\oplus\epsilon} \text{ and } B \cap C_n \subseteq A_{\oplus\epsilon}\} \\
d_n(A, B) &:= \min\{\tilde{d}_n(A, B), \operatorname{diam} C_n\} \\
d^!(A, B) &= \sum_{n=1}^{\infty} \alpha_n d_n(A, B)
\end{aligned}$$

a) Compute $d_n(B^2, \mathbb{Z}^2)$, where $B^2 := \{x \in \mathbb{R}^2 \mid \|x\| \leq 1\}$ is the closed unit ball, in $E = \mathbb{R}^2$ if $C_n = [-\pi, \pi]^2$ and if $C_n = [-0{,}9,\ 0{,}9]^2$.

b) For any $\delta > 0$ give an example of three sets $A, B, D \in \mathcal{F}$ with $\tilde{d}_n(A, B) < \delta$, $\tilde{d}_n(B, D) < \delta$ and $\tilde{d}_n(A, D) > 1$, where $E = \mathbb{R}^2$ and $C_n = [-1, 1]^2$.

c) Show that $d^! : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ is a semimetric.

We will show later that indeed $\lim_{j\to\infty} d^!(F_j, F) = 0$ is equivalent to the convergence of the sequence $(F_j)_{j\in\mathbb{N}}$ to F.