03.04.2017 · Th

Brush-Up-Stochastik
SoSe 17, Blatt 1
Musterlösung

---

① (a) $\Omega = \mathbb{N}_0 \times \mathbb{N}_0$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0,1,2,\ldots\}$

$= \{\omega = (\omega_1, \omega_2);\ \omega_1 \in \{0,1,2,\ldots\}, \omega_2 \in \{0,1,2,\ldots\}\}$

ω_1 : # geschossene Tore Mannschaft 1

ω_2 : # ——— " ——— 2

[$\omega_1 : \omega_2 \ \hat{=}$ Spielstand am Ende]

(b) $\Omega = \{\omega = (\omega_1, \omega_2);\ \omega_1 \in \{$"Kliniken Wissenschaftsstadt", "Staudinger Straße", "Botanischer Garten", "Uni-Süd", "Uni-West"$\}$, $\omega_2 \in \mathbb{N}_0\}$

ω_1 : Haltestelle

ω_2 : # Leute die an Haltest. ω_1 einsteigen

② A_k : Im k-ten Wurf fällt Kopf, $k = 1, 2, \ldots$

(a) $(A_1 \cap A_2 \cap A_3) \cup (A_1^c \cap A_2^c \cap A_3^c)$

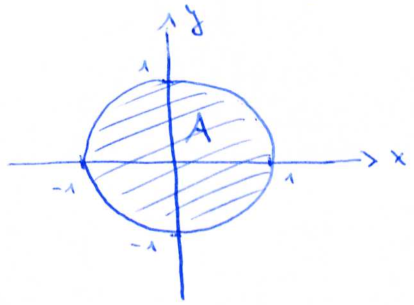
(b) $A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_{10} = \bigcup_{j=1}^{10} A_j$

(c) $(A_2 \cup A_4 \cup A_6 \cup \ldots) \cap (A_1^c \cap A_3^c \cap A_5^c \cap \ldots)$

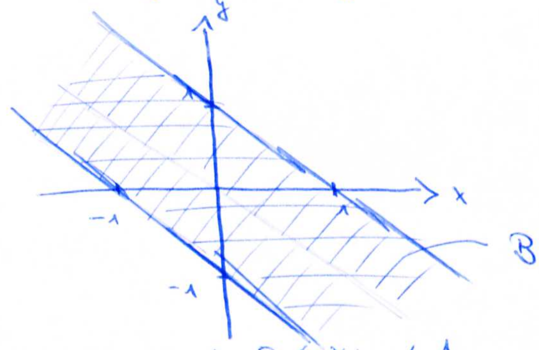
$= \left(\bigcup_{k=1}^{\infty} A_{2k}\right) \cap \left(\bigcap_{j=1}^{\infty} A_{2j-1}\right)$

3 $\Omega = \mathbb{R}^2$

$A = \{(x,y)\in\mathbb{R}^2 : x^2+y^2 \le 1\}$



$B = \{(x,y)\in\mathbb{R}^2 : |x+y| \le 1\}$



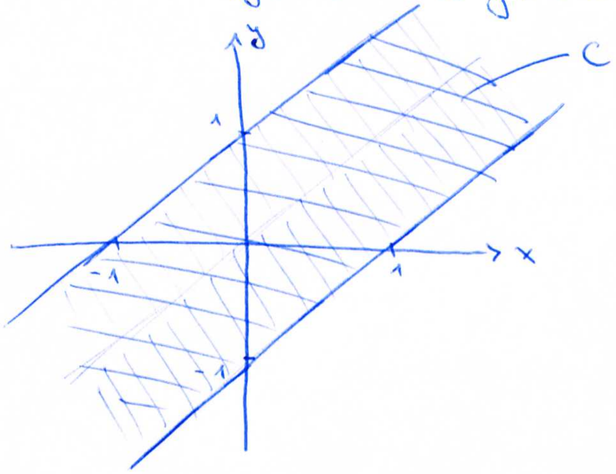
1.) $0 \le x+y \le 1$

$\Leftrightarrow -x \le y \le 1-x$

2.) $x+y \le 0 \Leftrightarrow y \le -x$

& $-(x+y) \le 1 \Leftrightarrow y \ge -x-1$

$C = \{(x,y)\in\mathbb{R}^2 : |x-y| \le 1\}$

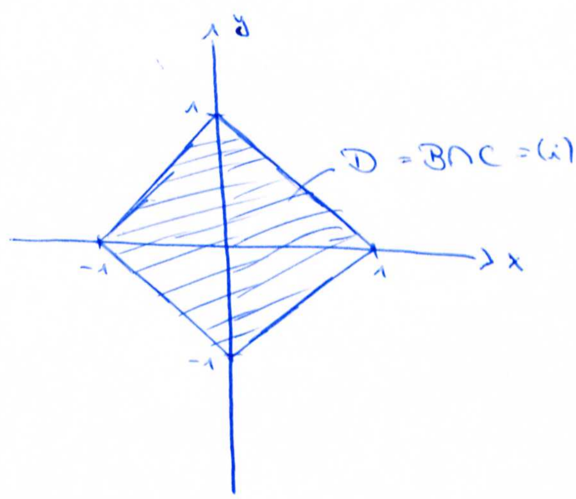


1.) $0 \le x-y \le 1 \Leftrightarrow x-1 \le y \le x$

2.) $x-y \le 0 \Leftrightarrow y \ge x$

& $-(x-y) \le 1 \Leftrightarrow y \le x+1$

$D = \{(x,y)\in\mathbb{R}^2 : |x|+|y| \le 1\}$



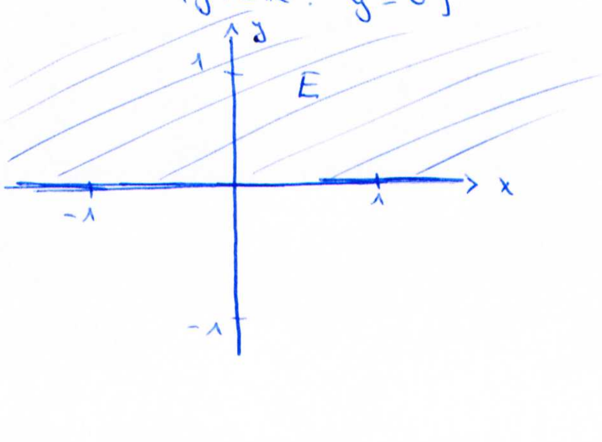
1.) $x, y \ge 0: \quad x+y \le 1$

2.) $x<0, y\ge 0: \quad -x+y \le 1$

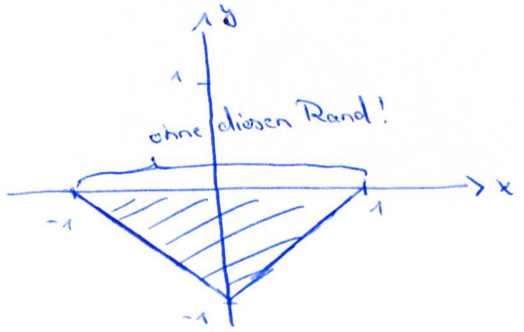
3.) $x \ge 0, y \le 0: \quad x-y \le 1$

4.) $x<0, y<0: \quad -x-y \le 1$

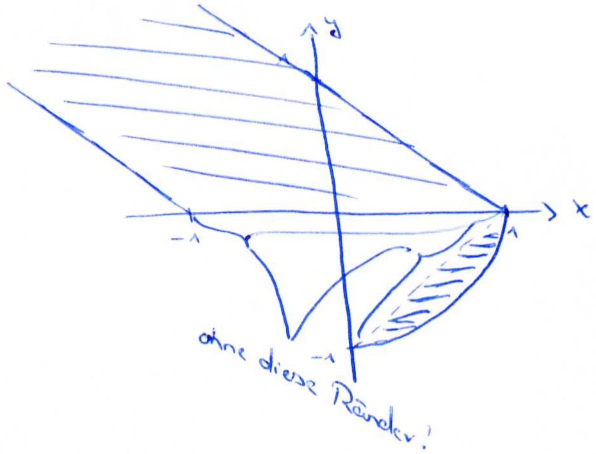
$E = \{(x,y)\in\mathbb{R}^2 : y \ge 0\}$



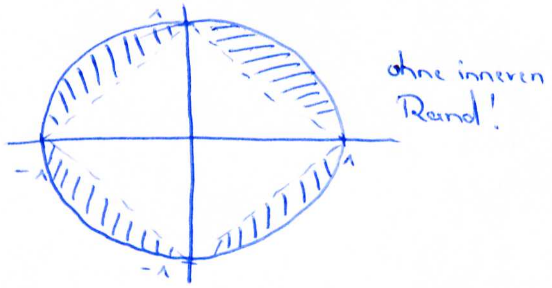
(ii) $D \setminus E$:



(iii) $(B \cap E) \cup (A \cap C^c)$:



(iv) $A \cap D^c$:



(v) $D \setminus A^c = D \cap A = D$ (da $A \supset D$)

(4) G7: Deutschland (D)
Frankreich (F)
Italien (I)
Japan (J)
Kanada (K)
UK
USA

(a) $\Omega = \{\omega = (\omega_1, \dots, \omega_7) ; \omega_1, \dots, \omega_7 \in \{D, F, I, J, K, UK, USA\}, \omega_i \neq \omega_j \ \forall i \neq j\}$

(b) (i) $|\Omega| = 7! = 5040$

(ii) $2 \cdot 5! = 240$

(iii) $5 \cdot 3! \cdot 4! = 720$

(5) (a) $\Omega = \{0,1,2,\dots,9\}^7 = \{(\omega_1,\dots,\omega_7)=\omega : \omega_1,\dots,\omega_7 \in \{0,1,\dots,9\}\}$.

$\Sigma = \mathcal{P}(\Omega)$

$P(A) = \dfrac{|A|}{|\Omega|}, \quad A \in \Sigma, \quad |\Omega| = 10^7$

(b) $A_1 = \{0000000\} \cup \{1234567\} \cup \{5252525\}$

$|A_1| = 3$

$\Rightarrow P(A_1) = \dfrac{|A_1|}{|\Omega|} = \dfrac{3}{10^7}$

(c) $A_2 = \{1,2,4,5,\dots,9\}^7$

$\Rightarrow |A_2| = 8^7 \quad \Rightarrow P(A_2) = \dfrac{|A_2|}{|\Omega|} = \dfrac{8^7}{10^7}$

$= \left(\dfrac{8}{10}\right)^7 \approx 0.21$

(d) $A_3 = \{(\omega_1,\dots,\omega_7) ; \omega_1,\dots,\omega_7 \in \{0,1,\dots,9\}, \omega_i \neq \omega_j \ \forall i \neq j\}$

$|A_3| = 10!$

$\Rightarrow P(A_3) = \dfrac{|A_3|}{|\Omega|} = \dfrac{10!}{10^7} = \dfrac{9!}{10^6} \approx 0.36$

(e) $A_4 = \{(\omega_1,\dots,\omega_7) : \exists! (i,j) \in \{1,\dots,7\}^2 \text{ mit } i \neq j \text{ und } \omega_i = \omega_j = 2\}$

$|A_4| = \binom{7}{2} 9^5 = 1240029$

$\Rightarrow P(A_4) = \dfrac{|A_4|}{|\Omega|} = \dfrac{1240029}{10^7} \approx 0.124$

(6) A, B, C Ereignisse, $P(A) = 0.3$, $P(B) = 0.4$,

$P(C) = 0.5$, $A \cap B = \emptyset$, $P(A \cap C) = P(A)P(C)$,

$P(B \mid C) = 0.5$

$$P(A \cup B \cup C) = \underbrace{P(A \cup B)}_{\substack{= P(A)+P(B) \\ \text{da } A \cap B = \emptyset}} + P(C) - \underbrace{P((A \cup B) \cap C)}_{\substack{= P((A \cap C) \cup (B \cap C)) \\ \text{wg. Distributivgesetz}}}$$

$$= P(A) + P(B) + P(C) - \underbrace{P((A \cap C) \cup (B \cap C))}_{\substack{= P(A \cap C) + P(B \cap C) \\ - \underbrace{P(A \cap B \cap C)}_{= 0 \text{ da } A \cap B = \emptyset}}}$$

$$\overset{\text{Unabh.}}{=} P(A) + P(B) + P(C) - P(A)P(C) - \underbrace{P(B \cap C)}_{= P(B \mid C) \cdot P(C)}$$

$$= 0.3 + 0.4 + 0.5 - 0.3 \cdot 0.5 - 0.5 \cdot 0.5$$

$$= 0.8$$

(7) G: "Kapitän in guter Form"

S: "Mannschaft gewinnt ein Spiel"

$P(G) = 0.7$, $P(G^c) = 0.3$

$P(S \mid G) = 0.75$, $P(S \mid G^c) = 0.4$

(a) $P(S) \overset{\text{tot.}}{\underset{\text{WKt.}}{=}} P(S \mid G)P(G) + P(S \mid G^c)P(G^c)$

$$= 0.75 \cdot 0.7 + 0.4 \cdot 0.3$$

$$= 0.645$$

(b) $P(G \mid S^c) \overset{\text{Bayes}}{=} \dfrac{P(S^c \mid G)P(G)}{P(S^c)}$

$$= \frac{(1-P(S \mid G))P(G)}{1-P(S)}$$

$$\overset{(a)}{=} \frac{0.25 \cdot 0.7}{0.355} \approx 0.49$$

8

K : "Kalteisen hat gepr."

R : " Ruhbach —"—"

H : " Halbherz —"—"

B : "2 der 3 Kand. best."

<u>Gesucht:</u> $P(K \mid B) \overset{\text{Bayes}}{=} \dfrac{P(B \mid K)P(K)}{P(B)}$

$P(K) = \frac{5}{20} = \frac{1}{4}$, $P(R) = \frac{8}{20} = \frac{2}{5}$

$P(H) = \frac{7}{20}$

$P(B \mid K) = 3 \cdot \left(\frac{1}{2}\right)^3 = \frac{3}{8}$

$P(B \mid R) = 3 \cdot \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 = \left(\frac{3}{4}\right)^3 = \frac{27}{64}$

$P(B \mid H) = 3 \cdot \frac{1}{3} \left(\frac{2}{3}\right)^2 = \frac{4}{9}$

$\overset{\text{tot.}}{\underset{\text{Wkt.}}{\Rightarrow}}$ $P(B) = P(B \mid K)P(K) + P(B \mid R)P(R) + P(B \mid H)P(H)$

$$= \frac{3}{8} \cdot \frac{1}{4} + \frac{27}{64} \cdot \frac{2}{5} + \frac{4}{9} \cdot \frac{7}{20}$$

$$\Rightarrow P(K \mid B) = \frac{\frac{3}{8} \cdot \frac{1}{4}}{\frac{3}{8} \cdot \frac{1}{4} + \frac{27}{64} \cdot \frac{2}{5} + \frac{4}{9} \cdot \frac{7}{20}} = \ldots$$

Zur Integration:

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{\sqrt{x-1}} \mathbb{1}_{(1,2]}(x)\,dx$

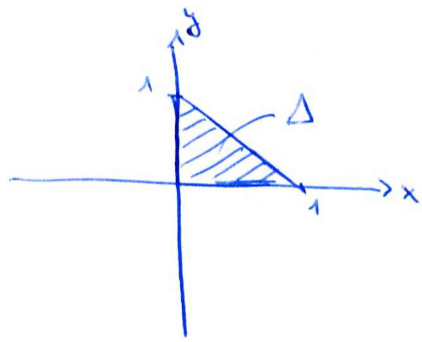
$\displaystyle = \int_1^2 \frac{1}{\sqrt{x-1}}\,dx = \left[2\sqrt{x-1}\right]_{x=1}^{2} = 2$

(b) $\displaystyle\int_{-\infty}^{\infty} x e^{-x} \mathbb{1}_{(0,\infty)}(x)\,dx$

$\displaystyle = \int_0^\infty x e^{-x}\,dx \overset{\text{P.I.}}{=} \overbrace{-xe^{-x}\Big|_{x=0}^{\infty}}^{=0} + \int_0^\infty e^{-x}\,dx$

$\displaystyle = \left[-e^{-x}\right]_0^\infty = 1$

(c) $\Delta = \{(x,y) \in [0,\infty) : x+y \leq 1\}$



$\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{x+y} \mathbb{1}_\Delta(x,y)\,dx\,dy$

$\displaystyle = \int_0^1 \int_0^{1-y} e^{x+y}\,dx\,dy$

$\displaystyle = \int_0^1 e^y \underbrace{\int_0^{1-y} e^x\,dx}_{= \left[e^x\right]_{x=0}^{1-y} = e^{1-y} - 1}\,dy$

$\displaystyle = \int_0^1 e - e^y\,dy = e - \underbrace{\int_0^1 e^y\,dy}_{= \left[e^y\right]_{y=0}^{1} = e-1} = 1$