04.04.2017

Brush-Up Stochastik

Blatt 2

Musterlösung

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(1) 102 Buchstaben, 35 Vok., 3 Umlaute, 62 Konson., 2 Joker

(a) A_1: 5 Kons. & 2 Vok.

$$P(A_1) = \frac{\binom{35}{2}\binom{62}{5}\binom{5}{0}}{\binom{102}{7}}$$

(b) A_2: mind. 1 Joker

$\hat{=}$ kein Joker

$$P(A_2) = 1 - P(\widetilde{A_2^c}) = 1 - \frac{\binom{100}{7}\binom{2}{0}}{\binom{102}{7}}$$

(c) A_3: kein Vokal

$$P(A_3) = \frac{\binom{67}{7}\binom{35}{0}}{\binom{102}{7}}$$

<u>Bem.:</u> Jeweils mit Hypergeom. Modell.

(2) E : Ü-Ei enthält Auto

A : Frau M. kauft bei Superm. A ein

B : ————— " ————— B "

C : ————— " ————— C "

D : ————— " ————— D "

Gegeben: $P(A) = 0.2$, $P(B) = 0.4$, $P(C) = 0.25$,

$P(D) = 0.15$

sowie $P(E|A) = 0.2$, $P(E|B) = 0.3$,

$P(E|C) = 0.1$, $P(E|D) = 0.05$

(a) $$P(E) \overset{\text{tot.}}{\underset{\text{Wkt.}}{=}} P(E|A)P(A) + P(E|B)P(B) + P(E|C)P(C) + P(E|D)P(D)$$

$$= 0.2 \cdot 0.2 + 0.3 \cdot 0.4 + 0.1 \cdot 0.25 + 0.05 \cdot 0.15$$

$$= 0.1925$$

(b) $$P(C|E) \overset{\text{Bayes}}{=} \frac{P(E|C)P(C)}{P(E)}$$

$$\overset{(a)}{=} \frac{0.1 \cdot 0.25}{0.1925} \approx 0.13$$

③ $\rho_i : \mathbb{R} \to \mathbb{R}$, $i \in \{1,2\}$

(a) $1 \overset{!}{=} \int_{-\infty}^{\infty} \frac{c}{\sqrt{x-1}} \mathbb{1}_{(1,2]}(x)\,dx$

$= c \cdot \int_1^2 \frac{1}{\sqrt{x-1}}\,dx \overset{\text{Blatt 1}}{\underset{\text{letzte Aufg.}}{=}} 2 \cdot c$

$\Leftrightarrow \quad c = \frac{1}{2}$

wg. $\frac{1}{2\sqrt{x-1}} \mathbb{1}_{(1,2]}(x) \geq 0 \quad \forall x \in \mathbb{R}$, ist

$\rho_1(x)$ für $c = \frac{1}{2}$ eine W-Dichte.

(b) $1 \overset{!}{=} \int_{-\infty}^{\infty} c e^{-|x|}\,dx$

$\underset{(*)}{=} c \cdot 2 \int_0^{\infty} e^{-x}\,dx = 2c\left[-e^{-x}\right]_{x=0}^{\infty}$

(*) wg. Symmetrie

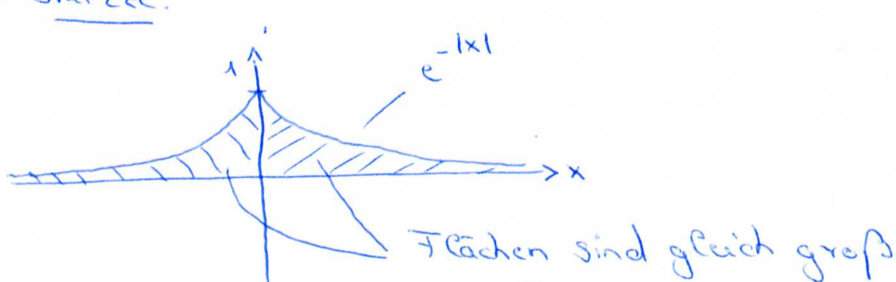
$= 2c$

$\Leftrightarrow \quad c = \frac{1}{2}$

wg. $\frac{1}{2} e^{-|x|} > 0 \quad \forall x \in \mathbb{R}$ ist $\rho_2(x)$

für $c = \frac{1}{2}$ eine W-Dichte.

(*) <u>Skizze:</u>



$\Rightarrow \int_{-\infty}^{\infty} e^{-|x|}\,dx = 2\int_0^{\infty} e^{-x}\,dx$

(4) $\rho_i : \mathbb{R} \to \mathbb{R}$, $i \in \{1, \dots, 4\}$

(a) $\rho_1(x) \geq 0 \quad \forall x \in \mathbb{R}$ und

$$\int_{-\infty}^{\infty} \rho_1(x)\,dx = \int_0^1 \frac{1}{2\sqrt{x}}\,dx = \left[\sqrt{x}\right]_{x=0}^1 = 1$$

$\Rightarrow$ ρ_1 ist eine W-Dichte.

$$F_1(x) = \int_{-\infty}^{x} \rho_1(t)\,dt$$

$$= \begin{cases} 0 & ; \quad x \leq 0 \\ \int_0^x \frac{1}{2\sqrt{t}}\,dt & ; \quad 0 < x \leq 1 \\ 1 & ; \quad x > 1 \end{cases}$$

$$= \begin{cases} 0 & ; \quad x \leq 0 \\ \sqrt{x} & ; \quad 0 < x \leq 1 \\ 1 & ; \quad x > 1 \end{cases}$$

(b) $\rho_2(x) \geq 0 \quad \forall x \in \mathbb{R}$, da $2 - \alpha > 0 \Leftrightarrow \alpha < 2$.

Außerdem:

$$\int_{-\infty}^{\infty} \rho_2(x)\,dx = (2-\alpha)\int_0^1 x^{-\alpha+1}\,dx$$

$$= (2-\alpha)\left[\frac{x^{-\alpha+2}}{-\alpha+2}\right]_{x=0}^1$$

$$= \left[x^{2-\alpha}\right]_{x=0}^1 = 1^{2-\alpha} = 1$$

$\Rightarrow$ ρ_2 ist eine W-Dichte.

(**)

(c) $\rho_3(x) \geq 0 \quad \forall x \in \mathbb{R}$ und

$$\int_{-\infty}^{\infty} \rho_3(x)\,dx = \alpha \int_0^{\infty} x^{\alpha-1} e^{-x^\alpha}\,dx \overset{\text{Subst.}}{=} \int_0^{\infty} e^{-y}\,dy = \left[-e^{-y}\right]_{y=0}^{\infty} = 1$$

$$y = x^\alpha$$

$$\frac{dy}{dx} = \alpha x^{\alpha-1}$$

$$\Leftrightarrow dx = \frac{dy}{\alpha x^{\alpha-1}}$$

$\Rightarrow$ ρ_3 ist eine W-Dichte.

$$F_3(x) = \int_{-\infty}^{x} \rho_3(t)\,dt$$

$$= \begin{cases} 0 & ; x \leq 0 \\ \underbrace{\int_0^x \alpha t^{\alpha-1} e^{-t^\alpha}\,dt}_{\overset{\text{wie}}{\underset{\text{oben}}{=}} \int_0^{x^\alpha} e^{-t}\,dt = \left[-e^{-t}\right]_{t=0}^{x^\alpha} = 1-e^{-x^\alpha}} & ; x > 0 \end{cases}$$

$$= \begin{cases} 0 & ; x \leq 0 \\ 1-e^{-x^\alpha} & ; x > 0 \end{cases}$$

(d) $\rho_4(x) \geq 0 \quad \forall x \in \mathbb{R}$, da $(x-\alpha)^2 \geq 0 \quad \forall x \in \mathbb{R}$.

Außerdem:

$$\int_{-\infty}^{\infty} \rho_4(x)\,dx = \frac{1}{\pi} \underbrace{\int_{-\infty}^{\infty} \frac{1}{1+(x-\alpha)^2}\,dx}$$

$$= \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$$

$$\overset{\text{Hinw.}}{=} \left[\arctan(x)\right]_{x=-\infty}^{\infty}$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

$$= \frac{1}{\pi}\pi = 1$$

$\Rightarrow$ ρ_4 ist eine W-Dichte.

$$F_4(x) = \int_{-\infty}^{x} \rho_4(t)\,dt = \int_{-\infty}^{x} \frac{1}{\pi} \frac{1}{1+(t-\alpha)^2}\,dt$$

$$\underset{\substack{\frac{dy}{dt}=1 \\ \Leftrightarrow dt = dy}}{\overset{y=t-\alpha}{=}} \int_{-\infty}^{x-\alpha} \frac{1}{\pi} \frac{1}{1+y^2}\,dy$$

$$\overset{\text{Hinw.}}{=} \frac{1}{\pi}\left[\arctan(y)\right]_{y=-\infty}^{x-\alpha}$$

$$= \frac{1}{\pi}\left(\arctan(x-\alpha) - \left(-\frac{\pi}{2}\right)\right)$$

$$= \frac{1}{\pi}\arctan(x-\alpha) + \frac{1}{2}$$

(**) Nachtrag:

$$F_2(x) = \int_{-\infty}^{x} \rho_2(t)\,dt$$

$$= \begin{cases} 0 & ; \; x \le 0 \\ \int_0^x (2-\alpha)t^{-\alpha+1}\,dt & ; \; 0 < x \le 1 \\ 1 & ; \; x > 1 \end{cases}$$

$$= \begin{cases} 0 & ; \; x \le 0 \\ x^{2-\alpha} & ; \; 0 < x \le 1 \\ 1 & ; \; x > 1 \end{cases}$$

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$$f_p(x) = \begin{cases} 2p & ; \; x \in \{-1, 1\} \\ p & ; \; x = 0 \\ 1-5p & ; \; x = 2 \\ 0 & ; \; \text{sonst} \end{cases}$$

(a) $f_p(x) \ge 0 \;\; \forall x \in \mathbb{R} \iff p \ge 0$

und $1-5p \ge 0 \iff p \le \frac{1}{5}$

d.h. $\forall p \in [0, \frac{1}{5}]$

Außerdem: $f_p(-1) + f_p(1) + f_p(0) + f_p(2)$

$= 2p + 2p + p + 1 - 5p = 1$

$\Rightarrow f_p$ ist eine Zähldichte falls $p \in [0, \frac{1}{5}]$

(b) X Zufallsv. mit Zähldichte f_p, $p \in [0, \frac{1}{5}]$

$r \in \mathbb{N}$:

$$\mathbb{E}[X^r] = (-1)^r 2p + 1^r 2p + 2^r (1-5p)$$

$$= \begin{cases} 4p + (1-5p)2^r & ; r \text{ gerade} \\ (1-5p)2^r & ; r \text{ ungerade} \end{cases}$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$$= 4p + (1-5p)4 - 4(1-5p)^2$$

$$= -16p + 4 - 4 + 40p - 100p^2$$

$$= 24p - 100p^2$$

6 (a) $X(\Omega) = \{ \frac{k\pi}{2} ; k = 0, \dots, 8 \}$, $P(X = x) = \frac{1}{9}$, $x \in X(\Omega)$,

$Y = \sin(X)$

$$\mathbb{E}[Y^r] = \mathbb{E}[(\sin(X))^r]$$

$$= \sum_{k=0}^{8} \left(\sin\left(\frac{k\pi}{2}\right)\right)^r \overbrace{P\left(X = \frac{k\pi}{2}\right)}^{= \frac{1}{9}}$$

$$= \frac{1}{9} \sum_{k=0}^{8} \underbrace{\left(\sin\left(\frac{k\pi}{2}\right)\right)}_{= \begin{cases} 0 & ; k \in \{0,2,4,6,8\} \\ 1 & ; k \in \{1,5\} \\ -1 & ; k \in \{3,7\} \end{cases}}{}^r$$

$$= \frac{1}{9}\left(2 + 2(-1)^r\right)$$

$$= \frac{2}{9}\left(1+(-1)^r\right) = \begin{cases} 0 & ; \; r \text{ ungerade} \\ \frac{4}{9} & ; \; r \text{ gerade} \end{cases}$$

$$\mathrm{Var}(Y) = \mathbb{E}[Y^2] - (\overbrace{\mathbb{E}[Y]}^{=0})^2$$

$$= \mathbb{E}[Y^2] = \frac{4}{9}$$

(b) $X(\Omega) = \{0, 1, \ldots, n\}$, $X \sim Bin(n, p)$, $p \in (0,1)$,

$Y = 2^X$

$$\mathbb{E}[Y^r] = \mathbb{E}[2^{rX}] = \sum_{k=0}^{n} 2^{rk} \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (2^r p)^k (1-p)^{n-k}$$

$$= (2^r p + 1 - p)^n = (1 + (2^r - 1)p)^n$$

$$\mathrm{Var}(Y) = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2$$

$$= (1 + (4-1)p)^n - (1 + (2-1)p)^{2n}$$

$$= (3p+1)^n - (p+1)^{2n}$$

(c) $X(\Omega) = \mathbb{N}_0$, $X \sim Poi(\lambda)$, $\lambda > 0$,

$Y = e^{2X}$

$$\mathbb{E}[Y^r] = \mathbb{E}[e^{2rX}] = e^{-\lambda} \sum_{k=0}^{\infty} e^{2rk} \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(e^{2r}\lambda)^k}{k!} \overset{\text{Hinw.}}{=} e^{-\lambda} \cdot e^{e^{2r}\lambda}$$

$$= e^{\lambda(e^{2r}-1)}$$

$$\mathrm{Var}(Y) = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2$$

$$= e^{\lambda(e^4-1)} - e^{2\lambda(e^2-1)}$$

(7) (a) $X \sim \mathrm{Exp}(\lambda)$, $\lambda > 0$, d.h.

$$p(x) = \lambda e^{-\lambda x} \mathbb{1}_{(0,\infty)}(x), \quad x \in \mathbb{R}.$$

$$\mathbb{E}[X] = \int_0^\infty x \lambda . e^{-\lambda x} dx$$

$$\overset{P.I.}{=} \underbrace{-x e^{-\lambda x} \Big|_{x=0}^{\infty}}_{=0} + \int_0^\infty e^{-\lambda x} dx$$

$$= \left[-\frac{1}{\lambda} e^{-\lambda x} \right]_{x=0}^{\infty} = \frac{1}{\lambda}$$

$$\mathbb{E}[X^2] = \int_0^\infty x^2 \lambda e^{-\lambda x} dx$$

$$\overset{P.I.}{=} \underbrace{-x^2 e^{-\lambda x} \Big|_{x=0}^{\infty}}_{=0} + 2 \underbrace{\int_0^\infty x e^{-\lambda x} dx}_{= \frac{1}{\lambda} \mathbb{E}[X] = \frac{1}{\lambda^2}}$$

$$= \frac{2}{\lambda^2}$$

$$\Rightarrow \mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

(b) $X \sim U(a,b)$, $0 < a < b < \infty$, d.h.

$$\rho(x) = \frac{1}{b-a} \mathbb{1}_{(a,b)}(x) \quad , x \in \mathbb{R}$$

$$\mathbb{E}[X] = \int_a^b x \frac{1}{b-a} \, dx = \frac{1}{b-a} \left[\frac{x^2}{2} \right]_{x=a}^{b}$$

$$= \frac{b^2 - a^2}{2(b-a)} = \frac{b+a}{2}$$

$$\mathbb{E}[X^2] = \int_a^b x^2 \frac{1}{b-a} \, dx = \frac{1}{b-a} \left[\frac{x^3}{3} \right]_{x=a}^{b}$$

$$= \frac{b^3 - a^3}{3(b-a)} = \frac{b^2 + ab + a^2}{3}$$

$$\Rightarrow \mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$$= \frac{b^2+ab+a^2}{3} - \left(\frac{b+a}{2}\right)^2$$

$$= \frac{b^2+ab+a^2}{3} - \frac{b^2+2ab+a^2}{4}$$

$$= \frac{b^2 - 2ab + a^2}{12} = \frac{(b-a)^2}{12}$$

(c) $X \sim N(\mu, \sigma^2)$, $\mu \in \mathbb{R}$, $\sigma^2 > 0$, d.h.

$$\rho(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) \quad , x \in \mathbb{R}$$

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) dx$$

$$\overset{y=x-\mu}{=} \int_{-\infty}^{\infty} (y+\mu) \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{y^2}{2\sigma^2}\right) dy$$

$\frac{dy}{dx} = 1$

$\Leftrightarrow dx = dy$

$$= \underbrace{\int_{-\infty}^{\infty} y \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{y^2}{2\sigma^2}\right) dy}_{\text{symmetrisch zum Koordinatenursprung} \Rightarrow \int \dots dy = 0} + \mu \underbrace{\int_{-\infty}^{\infty} \overbrace{\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{y^2}{2\sigma^2}\right)}^{(***)} dy}_{=1,\ \text{da (***) die W-Dichte der } N(0,\sigma^2)\text{ Vertlg.}}$$

$$= \mu$$

$$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$$

$$\overset{y=x-\mu}{=} \int_{-\infty}^{\infty} (y+\mu)^2 \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{y^2}{2\sigma^2}\right) dy$$

$$= \int_{-\infty}^{\infty} y^2 \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{y^2}{2\sigma^2}\right) dy$$

$$+ 2\mu \underbrace{\int_{-\infty}^{\infty} y \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{y^2}{2\sigma^2}\right) dy}_{=0 \text{ (wie oben!)}}$$

$$+ \mu^2 \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{y^2}{2\sigma^2}\right) dy}_{=1 \text{ (wie oben)}}$$

$$= \mu^2 + \int_{-\infty}^{\infty} y^2 \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{y^2}{2\sigma^2}\right) dy$$

$$= \mu^2 - \frac{\sigma^2}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \underbrace{y}_{u} \underbrace{\left(-\frac{y}{\sigma^2}\right) \exp\left(-\frac{y^2}{2\sigma^2}\right)}_{v'} dy$$

$$\Rightarrow v(y) = \exp\left(-\frac{y^2}{2\sigma^2}\right)$$

$$\overset{\text{P.I.}}{=} \mu^2 - \frac{\sigma^2}{\sqrt{2\pi\sigma^2}} \left(\underbrace{y \exp\left(-\frac{y^2}{2\sigma^2}\right) \Big|_{y=-\infty}^{\infty}}_{=0} - \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2\sigma^2}\right) dy \right)$$

$$= \mu^2 + \sigma^2 \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{y^2}{2\sigma^2}\right) dy}_{=1}$$

$$= \mu^2 + \sigma^2$$

$$\Rightarrow \operatorname{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$$= \mu^2 + \sigma^2 - \mu^2 = \sigma^2$$