Stochastic Simulations
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Exercise sheet 1 — till May 5, 2017

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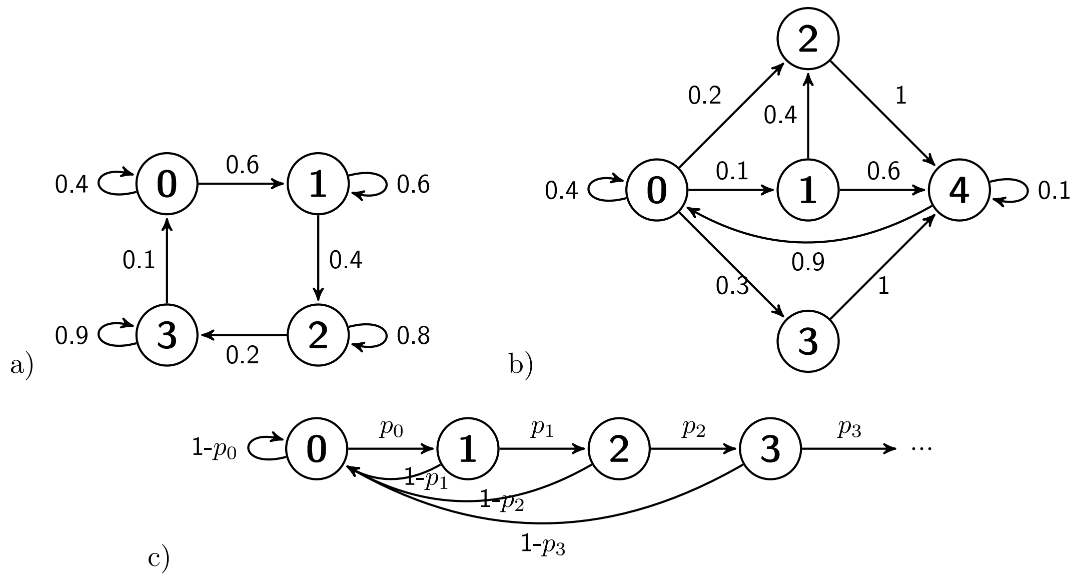
Theory (total – 13 points)

Exercise 1-1 (1+1+1 points)

Write down the transition matrices of the Markov chains, given by transition graphs:



Exercise 1-2 (1+1+1+1 points)

Draw the transition graphs of of the Markov chains, given by transition matrices:

$$(a)\begin{pmatrix} p_1 & p_2 & p_3 & \dots \\ 1 & 0 & 0 & \dots \\ 1 & 0 & 0 & \dots \\ & \dots & & \end{pmatrix}; \qquad (b)\begin{pmatrix} p_1 & p_2 & p_3 & p_4 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ & & \dots & & \end{pmatrix};$$

$$(c)\begin{pmatrix} p_1 & p_2 & p_3 & \dots \\ 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ & \dots & & \end{pmatrix}; \qquad (d)\begin{pmatrix} (1-p_1) & p_1 & 0 & 0 & \dots \\ (1-p_2) & 0 & p_2 & 0 & \dots \\ (1-p_3) & 0 & 0 & p_3 & \dots \\ & & \dots & & \end{pmatrix}.$$

Exercise 1-3 (1+2+2+1 points)

The transition matrix of a Markov chain $\{X_n, n \geq 0\}$ is

$$P = \begin{pmatrix} 0.1 & 0.2 & 0.7 \\ 0.3 & 0.4 & 0.3 \\ 0.5 & 0.4 & 0.1 \end{pmatrix}.$$

The initial distribution is: $\mathbf{P}(X_0 = 0) = 0.6$, $\mathbf{P}(X_0 = 1) = 0.3$, $\mathbf{P}(X_0 = 2) = 0.1$.

(1) Find the distribution of X_1.
(2) Find the probabilities $\mathbf{P}(X_2 = 0)$, $\mathbf{P}(X_1 = 0, X_2 = 1, X_3 = 2 | X_0 = 0)$,
$\mathbf{P}(X_1 = 0, X_3 = 0, X_4 = 2 | X_0 = 1)$, $\mathbf{P}(X_1 = 0, X_3 = 0, X_4 = 0, X_6 = 0, X_8 = 0 | X_0 = 0)$.
(3) Find the probabilities $\mathbf{P}(X_0 = 0, X_1 = 1)$,
$\mathbf{P}(X_1 = 0, X_2 = 1)$, $\mathbf{P}(X_4 = 0, X_3 = 1, X_1 = 0, X_2 = 2, X_5 = 0 | X_0 = 1)$.
(4) Find the distribution of the random variable $\tau_1 = \inf\{n \geq 0 | X_n \neq 0\}$.

Programming (total – 8 points)

Exercise 1-4 (4 points)

Let r.v. Y has the cumulative distribution function F and r.v. $X \sim U[0, 1]$. Prove that $F^{-1}(X) \stackrel{d}{=} Y$. Using this fact, write the `R` program to simulate random variables with the following distributions.
1. Gaussian with $\mu = 1$, $\sigma^2 = 1$,
2. Poisson(1),
3. Exp(1).

Simulate the samples of size $N = 1000$. Compare them with samples generated by standard R functions `rnorm, rpois, rexp.`

Exercise 1-5 (2+2 points)

A flea lives in a house with four dogs. Every day, it either stays where it is (with probability 0.7) or jumps (with probability 0.3) to one of the other dogs (selected uniformly).

(a) Write a `R` program to simulate this Markov chain using the initial distribution $\boldsymbol{\mu_0} = \boldsymbol{\delta_1}$. Run your simulation for $n = 365$ days and plot a histogram of the visited states.

(b) Calculate the distribution of the flea's position after 5, 10 and 365 days. Use the initial distribution from a).