Stochastic Simulations
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Exercise sheet 2 — till May 12, 2017

Please email your code to vitalii.makogin@uni-ulm.de till 8am, May 12

Theory (total – 11 points)

Exercise 2-1 (1+2 points)

Consider a Markov chain with state space $E = \{0, 1, 2\}$ and transition matrix

$$P = \begin{pmatrix} \frac{1}{8} & \frac{3}{8} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{1}{2} & \frac{1}{8} \end{pmatrix}.$$

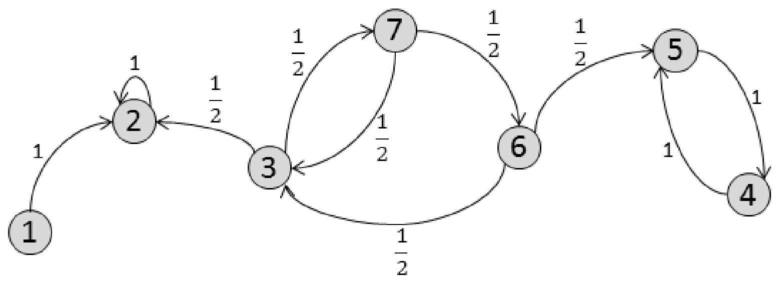
Find a random mapping representation of P using

(a) $Z \sim \mathrm{U}(0,1)$.

(b) $Z \sim \mathrm{Bin}(3, \frac{1}{2})$, i.e., Z binomial with parameters $n = 3$ and $p = \frac{1}{2}$.

Exercise 2-2 (2+1 points)

Consider a Markov chain whose transition matrix, P, is defined by the following graph



(a) Find the communicating classes of P. Which of these classes are closed?

(b) State if the transition matrix is irreducible and justify your answer.

Exercise 2-3 (3 points)

Suppose that a Markov chain $\{X_n, n \geq 0\}$ visits the set $A \subset \mathcal{X}$ infinitely many times with probability 1. Let τ_m be the mth moment. That is, $\tau_m = \inf\{k > \tau_{m-1} | X_k \in A\}$, where $\tau_{-1} \equiv 0$. Prove that $Y_n = X_{\tau_n}, n \geq 0$ is a Markov chain.

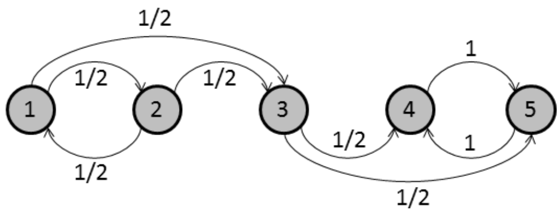
Exercise 2-4 (3 points)

Let $\{\varepsilon_n, n \in \mathbb{N}\}$ be a sequence of independent Bernoulli random variables, i.e., $\mathbf{P}(\varepsilon_n = 1) = p$, $\mathbf{P}(\varepsilon_n = -1) = 1 - p$. For which p is a sequence $\{X_n := \varepsilon_{n+1}\varepsilon_n, n \in \mathbb{N}\}$ a Markov chain?

Programming (total – 8 points)

Exercise 2-5 (4 points)

Consider the Markov chain $\{Y_n\}_{n\in\mathbb{N}}$ with initial distribution $\boldsymbol{\mu_0} = \boldsymbol{\delta_1}$ and the following transition graph.



Consider the following random variables:

(i) $\tau_1 = \inf\{n \geq 0 : Y_n + Y_{n+1} = 5\}$

(ii) $\tau_2 = \inf\{n \geq 0 : Y_{\lceil n/2 \rceil} \geq 4\}$

(iii) $\tau_3 = \sup\{n \geq 0 : Y_n \in \{1, 2, 3\}\}$

Write a R program to simulate $\{Y_n\}_{n\in\mathbb{N}}$ and estimate the expectation of τ_1, τ_2 and τ_3 if they are stopping times of $\{Y_n\}_{n\in\mathbb{N}}$. Use a sample size of at least $N = 10^3$.

Exercise 2-6 (4 points)

Consider the Markov chain $\{X_n\}_{n\in\mathbb{N}}$ with state space $\mathbb{Z}$, initial distribution $\mu_{\mathbf{0}} = \delta_{\mathbf{0}}$ and transition probabilities

$$p_{i,j} = \begin{cases} \frac{1}{4}, & \text{if } j = i - 1, \\ \frac{1}{2}, & \text{if } j = i, \\ \frac{1}{4}, & \text{if } j = i + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Simulate $\{X_n\}_{n\in\mathbb{N}}$ and draw one realization of the first 10 steps.