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Stochastic Simulations
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Exercise sheet 3 — till May 19, 2017

Theory (total – 12 points)

Exercise 3-1 (4 points)

Let $\{X_n\}_{n\in\mathbb{N}}$ be a random walk in $\mathbb{Z}$, $\mathbf{P}(X_{n+1} = i+1 | X_n = i) = p$, $\mathbf{P}(X_{n+1} = i-1 | X_n = i) = q = 1-p$, $i \in \mathbb{Z}$, where $p \in (0,1)$. Find the n-step transition matrix. For which p is the walk recurrent?

Exercise 3-2 (2+2 points)

Transition matrix P of a Markov chain is equal to

$$(a)\begin{pmatrix} 0.2 & 0.8 & 0 & 0 & 0 \\ 0.4 & 0.6 & 0 & 0 & 0 \\ 0.1 & 0.2 & 0.3 & 0.2 & 0.2 \\ 0 & 0 & 0 & 0.1 & 0.9 \\ 0 & 0 & 0 & 0.6 & 0.4 \end{pmatrix};(b)\begin{pmatrix} 0.1 & 0.2 & 0.3 & 0.1 & 0.1 & 0.2 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.2 & 0.4 \\ 0 & 0 & 0.4 & 0.6 & 0 & 0 \\ 0 & 0 & 0.3 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \end{pmatrix}.$$

1) Classify the states. 2) Find the stationary distributions.

Exercise 3-3 (4 points)

A total of N people live in two neighbouring cities, A and B. For some strange reason, exactly one person each month gets fed up with his or her city and moves to the neighbouring city. No one else moves. People are more likely to get sick of a city if there are more people in it. More precisely, consider the number of people in city A. Call this number k. Each month, the probability that someone moves from city A to the neighbouring city is $\frac{k}{N}$. The probability someone moves from city B to city A is the complementary probability, $\frac{N-k}{N}$.

Compute the stationary distribution of the Markov chain $(X_n)_{n\in\mathbb{N}}$ describing the number of people living in city A by solving the detailed balance equations. Classify the states. Find $\mathbf{E}_i\tau_i^F$.

Programming (total – 10 points)

Exercise 3-4 (6 points)

Write a R program to simulate the Markov chain $\{X_n\}_{n\in\mathbb{N}}$ defined in Exercise 3-4 for $N = 25$. Use the initial distribution $\boldsymbol{\mu_0} = \boldsymbol{\delta_0}$ and plot a histogram of the first 100 states, the first 1000 states and the first 10000 states.

For each $l = \overline{0, N}$ generate sample of size 1000 for τ_l^F with initial distribution $\boldsymbol{\delta_l}$. Estimate distributions of τ_l^F and the mean values. Compare with $\mathbf{E}_i \tau_i^F$.

Exercise 3-5 (4 points)

Write a R program to simulate the Markov chain $\{X_n\}_{n \in \mathbb{N}}$ defined in Exercise 3-2 b). Using the initial distribution $\boldsymbol{\mu_0} = \boldsymbol{\delta_1}$,
– plot a histogram of the first first 1000 states.
– build the samples of the first passage times τ_i^F, $i = \overline{1, 6}$. Use a sample size of at least $N = 10^3$. Estimate the distribution and mean values of τ_i^F, $i = \overline{1, 6}$.
Simulate $\{X_n\}_{n \in \mathbb{N}}$ and estimate $\mathbf{E}_i \tau_i^F$, $i = \overline{1, 6}$. Use a sample size of at least $N = 10^3$.