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Stochastic Simulations
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Exercise sheet 5 till June 2, 2017

Theory (total – 9 points)

Exercise 5-1 (2 points)

Prove that if $U \sim \mathcal{U}(0,1)$ and $n \in \mathbb{N}$, then $\lceil n \cdot U \rceil$ and $\lfloor n \cdot U \rfloor$ have (discrete) uniform distribution on $\{1, \dots, n\}$ and $\{0, \dots, n-1\}$, respectively.

Exercise 5-2 (4 points)

The acceptance-rejection algorithm introduced in the lecture works analogously for absolutely continuous distributions. Suppose we have a density f we want to sample from and a density g we can easily sample from and there is a constant $C > 0$ such that $Cg(x) \geq f(x)\ \forall x \in \mathbb{R}$. Then we can draw from f as follows:

- Draw X according to g and $U \sim \mathcal{U}(0,1)$ (independently).
- Accept X if $U \leq \frac{f(X)}{Cg(X)}$, reject otherwise.

Show that this approach leads to the desired result, i.e., show that

$$P\left(X \leq x \;\middle|\; U \leq \frac{f(X)}{Cg(X)}\right) = F(x),$$

where F is the cumulative distribution function of f.

Hint: Use the continuous version of the law of total probability, i.e., if X is an absolutely continuous random variable with density f, then

$$\mathbf{P}(A) = \int_{\mathbb{R}} \mathbf{P}(A \mid X = x) f(x) \mathrm{d}x.$$

Exercise 5-3 (3 points)

A random variable with density $g(y) = \sqrt{2/\pi} e^{-y^2/2} \mathbb{I}\{y \geq 0\}$ is to be simulated by rejection sampling. The candidate values are realizations from a $Exp(\lambda)$-distributed random variable with density $f(x) = \lambda e^{-\lambda x} \mathbb{I}\{x \geq 0\}$.

(a) Determine the smallest value c (with subject to $\lambda > 0$) such that $cf(y) \geq g(y)$.

(b) For which value of λ is the theoretical percentage of rejected samples minimal?

Programming (total – 11 points)

Exercise 5-4 (4 points)

Under the conditions Exercise 5-3, for $\lambda = 1$ write a R program to generate pseudorandom numbers according to the stated setup. Determine the theoretical percentage of rejected values and compare it to the empirical result for $n = 1000$ iterations.

Exercise 5-5 (4 points)

For each of the following densities, write down an algorithm based on the acceptance-rejection method to generate pseudorandom numbers according to each of the given distributions - call it G (with probability function $q = (q_1, \dots, q_{100})$ and density $g(y)$ respectively). Assume that the only available random number generator produces $U(0,1)$-pseudo random numbers.

(a) $q_j = a/j$, $j = 1, \dots, 100$ with $a = (\sum_{j=1}^{100} q_j)^{-1}$,

(b) $g(y) = \left(\frac{1}{10}y^2 + \frac{7}{15}\right)\mathbb{I}\{y \in (-1,1)\}$.

Hint: With regard to part (b): The auxiliary distribution F, from which candidates for the realizations of G are drawn, should be simple to generate, but not too far-off from the desired distribution.

Exercise 5-6 (3 points)

Write a R program to simulate random variable with the following distribution using the inversion method.

(a) Hypergeometric(m, n, k)

(b) Probability function is $q = (q_1, \dots, q_{50})$, where $q_j = a/j^2$, $j = 1, \dots, 50$ with $a = (\sum_{j=1}^{50} q_j)^{-1}$,