Stochastic Simulations
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Exercise sheet 7 till June 14, 2017

Theory (total – 12 points)

Exercise 7-1 (3 points)

Assume that $\{X_n\}_{n\in\mathbb{N}}$ is an aperiodic Markov chain, and all states communicate. Prove that for any $d \in \mathbb{N}$ a sequence $Y_n = X_{nd}, n \geq 0$ is an aperiodic Markov chain, and all states of $\{Y_n\}_{n\in\mathbb{N}}$ communicate.

Exercise 7-2 (3 points)

Suppose $\{X_n\}_{n\geq 0}$ is a Markov chain with $X_0 = i$. Denote by f_{ij} the probability of ever reaching j from i. Let N be the total number of visits made subsequently by the chain to the state j. Show that

$$\mathbf{P}(N = n) = \begin{cases} 1 - f_{ij}, & n = 0 \\ f_{ij} f_{jj}^{n-1}(1 - f_{jj}), & n \geq 1. \end{cases}$$

Exercise 7-3 (2 points)

Classify the states of the Markov chain with the state space $E = \{1, 2, 3, 4\}$ and transition matrix

$$\begin{pmatrix} 1/3 & 2/3 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 0 & 1/4 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Denote by $f_{ij}^{(n)} = \mathbf{P}(\tau_j^F = n | X_0 = i)$ the probability for the chain to make the first visit to a state j on the nth step given the chain started from the state i. Compute $f_{34}^{(n)}$ and deduce that the probability of ultimate absorption in state 4, starting from 3, equal $2/3$.

Exercise 7-4 (4 points)

Consider a Markov chain with state space $\mathcal{X} = \{1, 2\}$ and transition matrix

$$P = \begin{pmatrix} 1 - \alpha & \alpha \\ \beta & 1 - \beta \end{pmatrix},$$

where $\alpha, \beta \in (0, 1)$. Suppose you run two Markov chains, $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$, with transition matrix P independently, where $X_0 = 1$ and $Y_0 = 2$. Calculate the distribution of

$$\tau = \inf\{n \geq 0 : X_n = Y_n\}.$$

Programming (total – 11 points)

Exercise 7-5 (4 points), till June 16

Suppose n employees who come to work by car every day share n parking lots at their company. Every morning when they come to work, they park their cars randomly (uniformly) on the n parking lots.

(a) Write an R program to sample from the distribution of the cars on the parking lots using acceptance-rejection and the uniform distribution on $\{1, \ldots, n\}^n$ as proposal distribution.

Hint: You can use an array of length n to represent this, where the index is the number of the parking lot and the entry is the number of the employee occupying it.

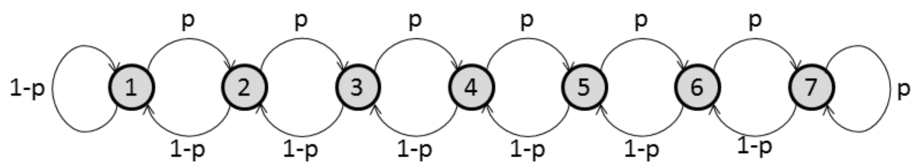
(b) Now assume that each employee, with probability $p = 0.1$, is ill and stays at home (they do this independently of one another). Adapt your algorithm to account for this using the uniform distribution on $\{0, \ldots, n\}^n$ for the proposals, where 0 indicates that a parking lot stays empty.

Hint: Simulate which employees are ill first, then use acceptance-rejection.

(c) Sample at least $N = 10^3$ times from the distributions in a) and b) and estimate the probability that the car of employee 1 stands next to the car of employee n, as well as the acceptance probability of your algorithm for $n = 3$, 5 and 10.

Exercise 7-6 (2 + 3 + 2 points) revised, till June 23

Consider two Markov chains $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ with the following transition graph.



(a) Write out a random mapping representation of $\{X_n\}_{n\in\mathbb{N}}$ using Bernoulli distributed random variables.

(b) Write an R program to simulate the random mapping coupling of $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ for $p = 0.6$, $X_0 = 1$ and $Y_0 = 7$ using a) and plot a realization of it up to the coupling time $\tau_{\mathsf{couple}} = \inf\{n \geq 0 : X_n = Y_n\}$.

(c) Run your program from b) at least $N = 10^4$ times and estimate the expected coupling time, $\mathbf{E}\tau_{\mathsf{couple}}$.