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Stochastic Simulations
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Exercise sheet 8 — till July 12, 2017

Theory (total – 7 points)

Exercise 8-1 (3 points)

An opera singer is due to perform a long series of concerts. Having a fine artistic temperament, she is liable to pull out each night with probability 1/2. Once this has happened she will not sing again until the promoter convinces her of his high regard. This he does by sending flowers every day until she returns. Flowers costing x thousand euros, $0 \geq x \geq 1$, bring about a reconciliation with probability x. The promoter stands to make 2500 Euro from each successful concert. How much should he spend on flowers?

Exercise 8-2 (4 points)

Each morning a student takes one of three books he owns from his shelf. The probability that he chooses book i is α_i, where $0 < \alpha_k < 1$, $k = 1, 2, 3$, and choices on successive days are independent. In the evening he replaces the book at the left-hand end of the shelf. If pn denotes the probability that on day n the student finds the books in order $1, 2, 3$ from the left to right, show that, irrespective of the initial arrangement of the books, pn converges as $n \to \infty$, and determine the limit.

Programming (total – 18 points)

Exercise 8-3 (3+2 points)

Consider the set Ω of all 5×5 matrices with values in $\{0, 1\}$ and the subset $A \subset \Omega$ of all such matrices which have no 1s directly next to each other in a row or column. This time, we do not want to draw uniformly from Ω but according to the distribution

$$\pi_T(x) = \frac{1}{Z_T} \exp(-\frac{1}{T}\varepsilon(x)) \quad \forall x \in A$$

with the energy function $\varepsilon(x)$

$$\varepsilon(x) = \begin{cases} \sum_{s \in \mathcal{S}} x_s \log(\lambda) & x \in \Omega \\ \infty & \text{otherwise,} \end{cases}$$

where $T = 1$ and $\lambda \in (0, \infty)$. Here $\mathcal{S}$ denotes the set of all sites in the matrix x.

(a) Write an R program to draw from π_T approximately, using the Metropolis algorithm.

(b) Run your program from a) for $\lambda = 0.5, 1$ and 1.5 and at least $N = 105$ steps and estimate the expected number of ones in the resulting random matrix. Hint: You can do so by averaging over all matrices in the Markov chain you run.

Exercise 8-4 (2+2+2+1 points)

Let us consider the generalized hard core model, which is formulated as follows. The graph is coloured by red and green such that no two neighboring vertices are red. Let the number of red vertices in a valid configuration $\mathbf{x}$ be weighted with a parameter $\lambda > 0$, i.e. one considers the probability mass function $\pi_\lambda = \{\pi_{\mathbf{x},\lambda}, \mathbf{x} \in E\}$ with

$$\pi_{\mathbf{x},\lambda} = \frac{\lambda^{n(\mathbf{x})}}{l_\lambda} \quad \forall \mathbf{x} \in E,$$

where $n(\mathbf{x})$ denotes the number of red vertices in $\mathbf{x}$ and $l_\lambda = \sum_{\mathbf{x} \in E} \lambda^{n(\mathbf{x})}$.

(a) Determine the conditional probability mass function $\pi_{1|\mathbf{x}(-v)}$ for $v \in V$.

(b) Construct a Gibbs sampler for the generalized hard core model.

(c) Let V be an 8×8-grid in two dimensions (use the 8-neighbourhood for your experiments). Let Y_λ be the random number of vertices having red color as the probability mass function π_λ is used. Estimate the mean value $\mathbb{E}(Y_\lambda)$ in the following way: Generate $n \cdot k = 10^6$ realizations ($n = 1000$) and put every n-th value into your sample. Provide estimates for $\lambda = 0.1, 1.0$ and 5.0.

(d) Plot two realizations of the hard core model after 10^6 steps for $\lambda = 1.0$.

Exercise 8-5 (2+2+2 points)

Consider the state space $E = \{\mathbf{x} = (x_1, \ldots, x_{N^2}) : x_m \in \{-1, +1\}\}$. The component x_m represents the m-th pixel of a $N \times N$-image, where $x_m = -1$ denotes a black pixel, and $x_m = +1$ a white one. The probability of a configuration $\mathbf{x} \in E$ is given by

$$\pi(\mathbf{x}) = \frac{1}{Z_T} \exp\left(\frac{1}{T} \sum_{(m,n)} x_m x_n\right) = \frac{1}{Z'_T} \exp\left(-\frac{2}{T} \#\mathbf{x}\right),$$

with constants $T, Z_T, Z'_T > 0$. The notation (m, n) says that one sum extends over all pairs of horizontal and vertical neighbours of the $N \times N$-grid, and $\#\mathbf{x}$ is the number of edges connecting differently coloured pixels (of the configuration $\mathbf{x}$).

(a) Show that the two representations of the discrete probability distribution $\boldsymbol{\pi} = (\pi(\mathbf{x}), \mathbf{x} \in E)$ are identical.

(b) Use a Metropolis-Hastings algorithm to construct a reversible Markov chain on E with limit distribution $\boldsymbol{\pi}$.

(c) What happens with the image if we decrease the parameter T?