Stochastic networks
Problem set 10
Due date: January 17, 2012

Exercise 1
Let \( \{N_B, B \in \mathcal{B}(\mathbb{R}^d)\} \) be a Poisson counting measure with intensity measure \( \mu \). Prove that for any Borel set \( B \subset \mathbb{R}^d \) with \( 0 < \mu(B) < \infty \) and any partition \( B = \bigcup_{i=1}^n B_i \) of \( B \) into pairwise disjoint Borel sets \( B_1, \ldots, B_n \subset \mathbb{R}^d \)

\[
\mathbb{P}(N_{B_1} = k_1, \ldots, N_{B_n} = k_n | N_B = k) = \frac{k!}{k_1! \cdots k_n!} \frac{\mu^{k_1}(B_1) \cdots \mu^{k_n}(B_n)}{\mu^k(B)}
\]

holds for all \( k, k_1, \ldots, k_n \geq 0 \) with \( k = k_1 + \ldots + k_n \).

Exercise 2

(a) Let \( \lambda > 0 \) and \( U_1, U_2, \ldots \) be a sequence of iid random variables with \( U_i \sim U([0,1]) \). Show that the random variables \( -\frac{\log U_1}{\lambda}, -\frac{\log U_2}{\lambda}, \ldots \) are iid with \( -\frac{\log U_i}{\lambda} \sim \text{Exp}(\lambda) \).

(b) Let \( X_1, X_2, \ldots \) be a sequence of iid random variables with \( X_i \sim \text{Exp}(\lambda) \). Show that \( Y \sim \text{Pois}(\lambda) \) where \( Y = \max\{k \geq 0 : X_1 + \ldots + X_k \leq 1\} \).

Exercise 3

(a) Construct an algorithm to generate realizations of a homogeneous Poisson point process \( \{N_B : B \in \mathcal{B}(\mathbb{R}^2)\} \) with intensity \( \lambda \) on a rectangular sampling window \( W \subset \mathbb{R}^2 \). Implement this algorithm in a suitable programming language.

(b) For \( r > 0 \) and \( \varphi \subset \mathbb{R}^2 \) locally finite write \( G(\varphi, r) \) for the graph with vertex set \( V = \varphi \), where an edge is drawn between \( x, y \in \varphi \) if and only if \( |x - y| \leq r \). Denote by \( N|_W \) the realization of a homogeneous Poisson point process in the rectangle \( W \). Write a computer program that determines all connected components of the graph \( G(N|_W, r) \) (here \( r > 0 \) is to be treated as a parameter).

(c) Write a computer program that calculates the number of vertices in each connected components of \( G(N|W_n, r) \) for \( \lambda = 1 \), \( r \in \{1, 2\} \) and

\[
W_n = [-n/2, n/2]^2 \text{ for } n \in \{50, 60, 70, 80, 90, 100\}
\]
(d) Compute the total edge-length of each connected component.

(e) Optimize your program so that it proceeds sequentially for each sampling window $W_n$ where the computation of the characteristics in (c) and (d) makes use of results obtained for smaller observation windows.

(f) What is the qualitative difference between the cases $r = 1$ and $r = 2$? (Consider the largest connected component, where largest means with respect to the number of vertices).

(g) For the two largest connected components compute (approximatively) the areas of influence, where the area of influence of a connected component $C = (V_C, E_C)$ is defined as \( \{ x \in \mathbb{R}^2 : |x - v| \leq r \text{ for some } v \in V_C \} \).

Hint. Use problems 1 and 2 to solve part (a)