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Stochastic networks

Problem set 10

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Exercise 1

Let $\{N_B,\, B \in \mathcal{B}(\mathbb{R}^d)\}$ be a Poisson counting measure with intensity measure μ. Prove that for any Borel set $B \subset \mathbb{R}^d$ with $0 < \mu(B) < \infty$ and any partition $B = \bigcup_{i=1}^n B_i$ of B into pairwise disjoint Borel sets $B_1, \dots B_n \subset \mathbb{R}^d$

$$\mathbb{P}(N_{B_1} = k_1, \dots, N_{B_n} = k_n \mid N_B = k) = \frac{k!}{k_1! \dots k_n!} \frac{\mu^{k_1}(B_1) \dots \mu^{k_n}(B_n)}{\mu^k(B)}$$

holds for all $k, k_1, \dots, k_n \geq 0$ with $k = k_1 + \dots + k_n$.

Exercise 2

(a) Let $\lambda > 0$ and $U_1, U_2, \dots$ be a sequence of iid random variables with $U_i \sim U([0,1])$. Show that the random variables $\frac{-\log U_1}{\lambda}, \frac{-\log U_2}{\lambda}, \dots$ are iid with $\frac{-\log U_i}{\lambda} \sim \mathrm{Exp}(\lambda)$.

(b) Let $X_1, X_2, \dots$ be a sequence of iid random variables with $X_i \sim \mathrm{Exp}(\lambda)$. Show that $Y \sim \mathrm{Pois}(\lambda)$ where $Y = \max\{k \geq 0 : \; X_1 + \dots + X_k \leq 1\}$.

Exercise 3

(a) Construct an algorithm to generate realizations of a a homogeneous Poisson point process $\{N_B : \; B \in \mathcal{B}(\mathbb{R}^2)\}$ with intensity λ on a rectangular sampling window $W \subset \mathbb{R}^2$. Implement this algorithm in a suitable programming language.

(b) For $r > 0$ and $\varphi \subset \mathbb{R}^2$ locally finite write $G(\varphi, r)$ for the graph with vertex set $V = \varphi$, where an edge is drawn between $x, y \in \varphi$ if and only if $|x - y| \leq r$. Denote by $N|_W$ the realization of a homogeneous Poisson point process in the rectangle W. Write a computer program that determines all connected components of the graph $G(N|_W, r)$ (here $r > 0$ is to be treated as a parameter).

(c) Write a computer program that calculates the number of vertices in each connected components of $G(N|_{W_n}, r)$ for $\lambda = 1$, $r \in \{1, 2\}$ and

$$W_n = [-n/2, n/2]^2 \text{ for } n \in \{50, 60, 70, 80, 90, 100\}$$

(d) Compute the total edge-length of each connected component.

(e) Optimize your program so that it proceeds sequentially for each sampling window W_n where the computation of the characteristics in (c) and (d) makes use of results obtained for smaller observation windows.

(f) What is the qualitative difference between the cases $r = 1$ and $r = 2$? (Consider the largest connected component, where *largest* means with respect to the number of vertices).

(g) For the two largest connected components compute (approximatively) the areas of influence, where the area of influence of a connected component $C = (V_C, E_C)$ is defined as $\{x \in \mathbb{R}^2 : |x - v| \leq r \text{ for some } v \in V_C\}$.

Hint. Use problems 1 and 2 to solve part (a)