Exercise 1

Let \( X = \{ X_n \}_{n \geq 1} \subset \mathbb{R}^d \) be a homogeneous Poisson point process with intensity \( \lambda \).

(a) Let \( U = \{ U_n \}_{n \geq 1} \) be a sequence of iid random variables (also independent of \( X \)) with \( U_i \sim U([0, 1]) \). Prove that \( Y = \{(X_n, U_n)\}_{n \geq 1} \subset \mathbb{R}^d \times [0, 1] \) is a Poisson point process and determine its intensity measure.

(b) Let \( p \in (0, 1) \) and let \( X' \subset X \) be defined by \( X' = \{ X_n \in X : U_n \leq p \} \). Prove that \( X' \subset \mathbb{R}^d \) is a homogeneous Poisson point process with intensity \( p\lambda \).

(c) Let \( a > 0 \) and let \( aX \subset \mathbb{R}^d \) be defined by \( aX = \{ a \cdot X_n \}_{n \geq 1} \). Prove that \( aX \) is a homogeneous Poisson point process with intensity \( a^{-d}\lambda \).

Exercise 2

Let \( X \subset \mathbb{R}^d \) be a homogeneous Poisson point process with intensity \( \lambda \). Furthermore let \( k \geq 1 \) and let \( h : \mathbb{R}^d \times \mathbb{N} \to [0, \infty) \) be a measurable function. Prove that

\[
\mathbb{E} \left( \sum_{\substack{x_1, \ldots, x_k \in X \\text{pw. disjoint}}} h(x_1, \ldots, x_k, X) \right) = \lambda^k \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \mathbb{E}(h(x_1, \ldots, x_k, X \cup \{x_1, \ldots, x_k\})) dx_1 \cdots dx_k
\]

Hint. Use induction on \( k \). For the case \( k = 1 \) you may assume (without loss of generality!) that there exists a bounded Borel set \( B \subset \mathbb{R}^d \) such that \( h(x, X) = 0 \) for \( x \notin B \) and such that \( h(x, \varphi) = h(x, \varphi \cap B) \). Then compute the expectation by conditioning on \( X(B) \).

Exercise 3

Let \( r, \lambda \geq 0 \), let \( X \subset \mathbb{R}^d \) be a homogeneous Poisson point process with intensity \( \lambda \) and write \( X_0 = X \cup \{o\} \). Furthermore denote by \( G(X_0, r) \) the geometric graph on the point process \( X_0 \) that was introduced in Exercise 3 of problem sheet 10.

(a) Compute the distribution of the degree of the vertex \( \{o\} \) in \( G(X_0, r) \).
(b) Compute the expected sum of lengths of all edges incident to \( \{o\} \) in \( G(X_0, r) \)

(c) Compute the expected (Euclidean) length of \([0, 1]^d \cap G(X, r)\)

(d) Compute the expected number of edges \( G(X, r) \) that intersect \([0, 1]^d\)

**Hint.** Problem 2 may be useful to solve part (c) and part (d)

**Exercise 4**

(a) Let \( X \subset \mathbb{R}^d \) be a homogeneous Poisson point process with intensity \( \lambda > 0 \). Prove that for all \( r > 0 \) the probability that \( G(X, r) \) is connected is equal to 0.

(b) Prove that for all \( r > 0 \) there exists \( C > 0 \) such that the probability that \( G(X^{(n)} \cap [0, n]^d, r) \) is connected tends to 1 as \( n \to \infty \), where \( X^{(n)} \subset \mathbb{R}^d \) is a homogeneous Poisson process with intensity \( \lambda_n = C \log(n) \).