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WS 2011/2012
January 17, 2012

Stochastic networks

Problem set 11

Due date: January 31, 2012

Exercise 1

Let $X = \{X_n\}_{n\geq 1} \subset \mathbb{R}^d$ be a homogeneous Poisson point process with intensity λ.

(a) Let $U = \{U_n\}_{n\geq 1}$ be a sequence of iid random variables (also independent of X) with $U_i \sim \mathrm{U}([0,1])$. Prove that $Y = \{(X_n, U_n)\}_{n\geq 1} \subset \mathbb{R}^d \times [0,1]$ is a Poisson point process and determine its intensity measure.

(b) Let $p \in (0,1)$ and let $X' \subset X$ be defined by $X' = \{X_n \in X : U_n \leq p\}$. Prove that $X' \subset \mathbb{R}^d$ is a homogeneous Poisson point process with intensity $p\lambda$.

(c) Let $a > 0$ and let $aX \subset \mathbb{R}^d$ be defined by $aX = \{a \cdot X_n\}_{n\geq 1}$. Prove that aX is a homogeneous Poisson point process with intensity $a^{-d}\lambda$.

Exercise 2

Let $X \subset \mathbb{R}^d$ be a homogeneous Poisson point process with intensity λ. Furthermore let $k \geq 1$ and let $h : \mathbb{R}^d \times \mathbb{N} \to [0,\infty)$ be a measurable function. Prove that

$$\mathbb{E}\left(\sum_{\substack{x_1,\dots,x_k \in X \\ x_1,\dots,x_k \text{ pw. disjoint}}} h(x_1,\dots,x_k,X) \right) = \lambda^k \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \mathbb{E}(h(x_1,\dots,x_k, X \cup \{x_1,\dots,x_k\}))dx_1 \cdots dx_k$$

Hint. Use induction on k. For the case $k = 1$ you may assume (without loss of generality!) that there exists a bounded Borel set $B \subset \mathbb{R}^d$ such that $h(x,X)$ is 0 for $x \notin B$ and such that $h(x,\varphi) = h(x, \varphi \cap B)$. Then compute the expectation by conditioning on $X(B)$.

Exercise 3

Let $r, \lambda \geq 0$, let $X \subset \mathbb{R}^d$ be a homogeneous Poisson point process with intensity λ and write $X_0 = X \cup \{o\}$. Furthermore denote by $G(X_0, r)$ the geometric graph on the point process X_0 that was introduced in Exercise 3 of problem sheet 10.

(a) Compute the distribution of the degree of the vertex $\{o\}$ in $G(X_0, r)$

(b) Compute the expected sum of lengths of all edges incident to $\{o\}$ in $G(X_0, r)$

(c) Compute the expected (Euclidean) length of $[0,1]^d \cap G(X, r)$

(d) Compute the expected number of edges $G(X, r)$ that intersect $[0,1]^d$

Hint. Problem 2 may be useful to solve part (c) and part (d)

Exercise 4

(a) Let $X \subset \mathbb{R}^d$ be a homogeneous Poisson point process with intensity $\lambda > 0$. Prove that for all $r > 0$ the probability that $G(X, r)$ is connected is equal to 0.

(b) Prove that for all $r > 0$ there exists $C > 0$ such that the probability that $G(X^{(n)} \cap [0, n]^d, r)$ is connected tends to 1 as $n \to \infty$, where $X^{(n)} \subset \mathbb{R}^d$ is a homogeneous Poisson process with intensity $\lambda_n = C \log(n)$.