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Stochastic networks

Problem set 12

Due date: February 9, 2012

Exercise 1

Prove the formal correctness of the acceptance-rejection algorithm on page 58f. of the lecture notes.

Exercise 2

Let $X \subset \mathbb{R}^2$ be a homogeneous Poisson process of intensity $\lambda < 1/\pi$ and consider the connected component C_o of o in the graph $G(X \cup \{o\}, 1)$. Formally construct a coupling between C_o and a subcritical Galton-Watson branching process $\{Z_n\}_{n\geq 1}$ whose offspring distribution is given by $\mathrm{Poi}(\lambda\pi)$ that satisfies $|C_o| \leq \sum_{n\geq 1} Z_n$.

Exercise 3

Let $g : [0, \infty) \to [0, 1]$ be a measurable non-increasing function such that $\int_{\mathbb{R}^2} g(|x|)dx < \infty$. Let $X = \{X_n\}_{n\geq 1}$ be a homogeneous Poisson process in $\mathbb{R}^2$ of intensity $\lambda > 0$ and let $U = \{U_{n,m}\}_{\substack{m,n\geq 1\\ m<n}}$ be a sequence of iid random variables that are independent of X and such that $U_{m,n} \sim \mathrm{U}([0, 1])$. Denote by $G(X, g)$ the geometric graph on the vertex set X and where for $m < n$ the points $X_n, X_m \in X$ are connected by an edge in $G(X, g)$ iff $U_{m,n} < g(|X_n - X_m|)$. Prove that $G(X, g)$ does not percolate if $\lambda \int_{\mathbb{R}^2} g(|x|)dx < 1$.

Hint. Try to adapt the proof from the lecture.

Exercise 4

Let $d \geq 2$ and let $X \subset \mathbb{R}^d$ be a homogeneous Poisson process of intensity λ. For $k \geq 1$ denote by $\widetilde{G}(X, k)$ the directed graph with vertex set X and where an edge is drawn from x to y if y is one of the k nearest neighbors of x in X. Furthermore denote by $G(X, k)$ the (undirected) graph where an edge is drawn between x and y if there exists a directed edge from x to y in $\widetilde{G}(X, k)$ or if there exists a directed edge from y to x in $\widetilde{G}(X, k)$. Compute explicit integral expressions for the following quantities and compute the integrals where possible.

(a) the expected number of edges pointing to the origin in the graph $\widetilde{G}(X \cup \{o\}, k)$

(b) the expected degree of o in $G(X \cup \{o\}, k)$

(c) $\mathbb{E}(\nu_1(G(X, k) \cap [0, 1]^d))$

Hint. Try to obtain the following expressions. Here the following notations are used

- ω_d denotes the surface area of the $d-1$-dimensional unit sphere
- κ_d denotes the volume of the d-dimensional unit ball
- $a_d = \nu_d(B_1(0) \cap B_1(e_1))$
- $b_d = \nu_d(B_1(0) \setminus B_1(e_1))$

(where ω_d resp. κ_d denote the surface area of the $d-1$-dimensional unit sphere resp. the volume of the d-dimensional unit ball)

(a) $\omega_d \int_0^\infty s^{d-1} \exp(-\kappa_d s^d) \sum_{i=0}^{k-1} (\kappa_d s^d)^i / i! ds$

(b)

$$k + \omega_d \int_0^\infty s^{d-1} \exp(-\kappa_d s^d) \sum_{i=0}^{k-1} (\kappa_d s^d)^i / i! ds$$
$$- \omega_d \int_0^\infty s^{d-1} \sum_{\ell=0}^{k-1} \exp(-a_d s^d)(a_d s^d)^\ell / \ell! \left(\sum_{m=0}^{k-1-l} \exp(-b_d s^d)(b_d s^d)^m / m! \right)^2 ds$$

(c)

$$\lambda^{(d-1)/d} \omega_d \int_0^\infty s^{2d-1} \exp(-\kappa_d s^d) \sum_{i=0}^{k-1} (\kappa_d s^d)^i / i! ds$$
$$- \lambda^{(d-1)/d} \omega_d \int_0^\infty s^{2d-1} \sum_{\ell=0}^{k-1} \exp(-a_d s^d)(a_d s^d)^\ell / \ell! \left(\sum_{m=0}^{k-1-l} \exp(-b_d s^d)(b_d s^d)^m / m! \right)^2 ds$$