Prof. Dr. V. Schmidt
C. Hirsch

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# Stochastic networks

## Problem set 1

Due date: October 25, 2011

In all problems you may assume that $V \subset \mathbb{R}^d$ is countable.

**Exercise 1**

Let $G = (V, E)$ be a locally finite graph (not necessarily connected). Let $\pi = \langle v_1, v_2, \ldots \rangle$ be a sequence of vertices in $G$ such that $v_i$ and $v_{i+1}$ are connected by an edge for all $i \geq 1$ and that has the property that the set $\{v_1, v_2, \ldots\}$ is infinite, but any vertex $v_i$ is only visited a finite number of times. Show that $G$ contains an infinite *self-avoiding* path (i.e. a path where no vertex is visited more than once). Also show that this statement is no longer true, if we remove **the local finiteness assumption and** the assumption that any vertex $v_i$ is only visited a finite number of times.

**Exercise 2**

Let $G = (V, E)$ be a connected and locally finite graph and let $p \in [0, 1]$. Prove that

(a) $\exists x \in V : \theta_x(p) > 0 \iff \forall x \in V : \theta_x(p) > 0$

(b) $\exists x \in V : \chi_x(p) < \infty \iff \forall x \in V : \chi_x(p) < \infty$

**Exercise 3**

Let $V = \{(x, 0) : x \in \mathbb{Z}\} \cup \{(0, y) : y \in \mathbb{Z}\}$ and $E = \{\{(x, 0), (x+1, 0)\} : x \in \mathbb{Z}\} \cup \{\{(0, y), (0, y+1)\} : y \in \mathbb{Z}\}$. Prove that for $G = (V, E)$ we have $p_c = p_{ec} = 1$.

**Exercise 4 ***

Let $G = (V, E)$ be a connected and locally-finite graph. In the lecture we discussed *bond*-percolation, i.e. we studied the probability space $(\{0, 1\}^E, (2^{\{0,1\}})^{\otimes E}, \mathbb{P}^b_{\mathbf{p}})$, for $\mathbf{p} \in [0, 1]^E$. Here $\mathbb{P}^b_{\mathbf{p}}$ is the probability measure introduced in the lecture which is determined by

$$\mathbb{P}^b_{\mathbf{p}}(C(F, z)) = \prod_{f \in F : z_f = 1} p_f \prod_{f \in F : z_f = 0} (1 - p_f),$$

for every finite set $F \subset E$ and every $z \in \{0, 1\}^F$. In *site*-percolation one does not study probability measures on configurations of edges, but on configurations of vertices. To be more precise: a site percolation model is a probability space $(\{0, 1\}^V, (2^{\{0,1\}})^{\otimes V}, \mathbb{P}^s_{\mathbf{p}})$ for $\mathbf{p} \in [0, 1]^V$. In fact site percolation is in a sense more general than bond percolation:

Denote by $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ the *line graph* of $G$. That is, $\widetilde{V} = E$ and $e, f \in \widetilde{V}$ are connected by an edge in $\widetilde{G}$ if and only if there exists a vertex $v \in V$ that is adjacent both to $e$ and to $f$. Note that any vector of probabilities $\mathbf{p} \in [0,1]^E$ defines a site percolation model $(\{0,1\}^{\widetilde{V}}, (2^{\{0,1\}})^{\otimes \widetilde{V}}, \widetilde{\mathbb{P}}^s_{\mathbf{p}})$. Prove that

$$\mathbb{P}^b_{\mathbf{p}}(\exists \text{ infinite activated component in } G) = \widetilde{\mathbb{P}}^s_{\mathbf{p}}(\exists \text{ infinite activated component in } \widetilde{G}).$$

Also make a drawing of the line graph of $\mathbb{Z}^2$!