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Stochastic networks

Problem set 2

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Exercise 1

Let $X_1, X_2, \ldots : \Omega \to \mathbb{R}$ be a sequence of iid random variables defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Furthermore write $\mathcal{A}_n = \sigma(X_1, \ldots, X_n)$ and $\mathcal{A}_\infty = \sigma(X_1, X_2, \ldots)$ and define $\mathcal{M} = \{A \in \mathcal{A}_\infty : \exists A_1, A_2, \ldots \in \bigcup_{n=1}^\infty \mathcal{A}_n : \lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(A) \text{ and } \lim_{n\to\infty} \mathbb{P}(A_n \cap A) = \mathbb{P}(A)\}$. Prove that $\mathcal{M}$ is a monotone class containing the algebra $\bigcup_{n=1}^\infty \mathcal{A}_n$.

Hint: Let $A_1, A_2, \ldots \in \bigcup_{n=1}^\infty \mathcal{A}_n$ and $A = \bigcup_{n=1}^\infty A_n$. First choose $A_{i,j} \in \bigcup_{n=1}^\infty \mathcal{A}_n$ with $\lim_{j\to\infty} \mathbb{P}(A_{i,j}) = \mathbb{P}(A_i)$ and $\lim_{j\to\infty} \mathbb{P}(A_{i,j} \cap A_i) = \mathbb{P}(A_i)$. Then try to use estimates of the form $\left|\mathbb{P}(A) - \mathbb{P}(A_{k,f(k)})\right| \le |\mathbb{P}(A) - \mathbb{P}(A_k)| + \left|\mathbb{P}(A_k) - \mathbb{P}(A_{k,f(k)})\right|$ for a suitable function f!

Exercise 2

Let $k \ge 3$, let $G = (V, E)$ be a k-regular tree and let $G' = (V', E')$ be a $k-1$-branching tree (see problem 6). Prove that for all $p \in [0, 1]$ we have $1 - \theta(p, G) = (1 - p\theta_0(p, G'))^k$. Furthermore show that $\theta_0(p, G')$ satisfies the equation $1 - \theta_0(p, G') = (1 - p\theta_0(p, G'))^{k-1}$.

Exercise 3

Denote by $\widetilde{V} = \bigcup_{n=0}^\infty \{0, 1\}^n$ the set of all finite binary sequences and write $V = \widetilde{V} \times \{a, b\}$. Now define the graph $G = (V, E)$ with $E = \{\{(v, a), (v, b)\} : v \in \widetilde{V}\} \cup \{\{(v, b), (c(v, 0), a)\} : v \in \widetilde{V}\} \cup \{\{(v, b), (c(v, 1), a)\} : v \in \widetilde{V}\}$. Here we write $c(v, \varepsilon) = (\varepsilon_1, \ldots, \varepsilon_m, \varepsilon)$ for all $\varepsilon \in \{0, 1\}$ and $v = (\varepsilon_1, \ldots, \varepsilon_m) \in \widetilde{V}$. First draw a picture of G and then prove that $p_c = p_{ec} = 1/\sqrt{2}$.

Hint: Try to adopt the proof of Theorem 2.4!

Exercise 4

Construct a countable, connected and locally finite graph $G = (V, E)$ with $p_{ec} = 0$ and $p_c = 1$.

Hint: Start with the graph $\mathbb{Z}_{\ge 1}$. Now connect each node $n \in \mathbb{Z}_{\ge 1}$ with a separate complete graph on 2^{n^2} vertices.

Exercise 5 *

Let $G = (V, E)$ be a countable, locally finite, but not necessarily connected graph. On any such graph, we may consider Bernoulli bond percolation (i.e. the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ defined in the lecture notes, page 6) and define $\widetilde{p_c}(G) = \inf(p \in [0,1] : \mathbb{P}_p(\Xi_p \text{ contains an infinite component} > 0))$. Prove that for all $p \in (p_c, 1]$ we have $\widetilde{p_c}(\Xi_p) = \widetilde{p_c}(G)/p$ a.s. Conclude that there exists a countable graph G with $\widetilde{p_c}(G) = 1/\pi$.

Hint: One part of the problem is to prove that if $q > \widetilde{p_c}(G)/p$, then for $\mathbb{P}_p$-almost every $\omega \in \{0,1\}^E$ there exists an infinite component in $\Xi_q(\Xi_p(\omega))$ with positive probability. Try to interpret $\Xi_q(\Xi_p(\omega))$ as a realization of $\Xi_{pq}(G)$ and note that for $pq > \widetilde{p_c}(G)$ the graph $\Xi_{pq}(G)$ contains an infinite component with probability 1. The following construction of Ξ_p and Ξ_q may be useful. Let $(U_e)_{e \in E}$ and $(\widetilde{U}_e)_{e \in E}$ be independent families of iid random variables with $U_e, \widetilde{U}_e \sim U([0,1])$. Define the family $(\widetilde{\widetilde{U}}_e)_{e \in E}$ as

$$\widetilde{\widetilde{U}}_e = \begin{cases} p \cdot \widetilde{U}_e \text{ if } U_e < p \\ U_e \text{ if } U_e > p \end{cases}$$

Prove that $(\widetilde{\widetilde{U}}_e)_{e \in E}$ are iid and $\widetilde{\widetilde{U}}_e \sim U([0,1])$ and try to work with $\Xi_p = \{e \in E : U_e < p\}$ and $\Xi_q(\Xi_p) = \{e \in E : \widetilde{\widetilde{U}}_e < pq\}$.

Exercise 6 *

Let $k \geq 3$, let $1/(k-1) < p < 1$ and let $G = (V, E)$ be a k-regular tree. Consider Bernoulli bond percolation (i.e. the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ defined in the lecture notes, page 6) on G and denote by N the number of infinite clusters in Ξ_p. Prove that $N = \infty$ a.s.

Hint: Proceed by contradiction, so assume that $\mathbb{P}(N = \infty) < 1$. Let $\ell = \inf(n \geq 0 : \mathbb{P}(N = n) > 0)$. The first step is to deduce that this would imply $\ell = 1$. As in the proof of $p_c = 1$ for k-regular trees, it is convenient to consider instead the k-branching tree $G' = (V', E')$. Recall that this can be defined by $V' = \bigcup_{n=0}^{\infty} \{0, 1, \ldots, k-1\}^n$ and $E = \bigcup_{i=0}^{k-1} \{\{v, c(v,i)\} : v \in V'\}$. Here we write $c(v, \varepsilon) = (\varepsilon_1, \ldots, \varepsilon_m, \varepsilon)$ for all $\varepsilon \in \{0, 1, \ldots, k-1\}$ and $v = (\varepsilon_1, \ldots, \varepsilon_m) \in V'$. For $i = 0, 1, \ldots, k-1$ define the subsets $V_i = \{(i, \varepsilon_1, \ldots, \varepsilon_m) : m \geq 0, \varepsilon_1, \ldots, \varepsilon_m \in \{0, 1, \ldots, k-1\}\}$ and consider the subgraphs G_i of G generated by V_i. Observe that each G_i is again a k-branching tree and prove that $N \geq N_0 + N_1 + \ldots N_{k-1} - (k-1)$, where N_i denotes the number of infinite activated components in G_i. Show that $N_0 + N_1 + \ldots + N_{k-1} \geq k \cdot \ell$ with probability 1 and deduce that $\ell = 1$. Finally derive and use the estimate $\mathbb{P}(N = 1) \leq p^k \mathbb{P}(N = 1)^k$ to obtain a contradiction.