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WS 2011/2012
November 22, 2011

Stochastic networks

Problem set 5

Due date: November 29, 2011

Exercise 1

Let $n \geq 1$ and let $(\Omega_n, \mathcal{A}_n, \mathbb{P}_n)$ be the probability space defined in the lecture. Furthermore, for $1 \leq k \leq n$, let $X_k : \Omega_n \to \{0, 1\}$ denote the projection on the k-th component. Which of the following events are increasing, decreasing, neither of the two?

(a) $|\{k : X_k = 1\}| = 5$

(b) $\forall 1 \leq k \leq n : (X_k = 1 \Rightarrow \forall j : (1 \leq j \leq k-1 \Rightarrow X_j = 0))$

(c) A_0, where $A \in \mathcal{A}_n$ is decreasing

Hint. The answer may depend on the value of n.

Exercise 2

Let $n \geq 1$ and let $(\Omega_n, \mathcal{A}_n, \mathbb{P}_n)$ be the probability space defined in the lecture. Let $f, g : \Omega_n \to [0, \infty)$ be bounded increasing functions (i.e. $f(\omega) \leq f(\omega')$ for $\omega \leq \omega'$).

(a) show that $\mathbb{E}(fg) \geq \mathbb{E}(f)\mathbb{E}(g)$.

(b) use part (a) to give a more conceptual proof of exercise 2 in problem set 1.

Exercise 3

Consider Bernoulli bond percolation on $\mathbb{Z}^d$. Prove that the function $\theta(p)$ is continuous from the right.

Hint. The relation $\theta(p) = \lim_{n\to\infty} \mathbb{P}_p(o \to \partial B(n))$ may be useful, where $\partial B(n)$ is the set of vertices in $\mathbb{Z}^d$ that can be reached from the origin o in at most n but not in $n-1$ steps. Furthermore you may use (without proof) that any decreasing limit of continuous functions is upper semicontinuous.

Exercise 4

Consider Bernoulli bond percolation in $\mathbb{Z}^2$ and let $p > p_c$. Denote by A_n the event that there exists an activated vertical crossing of the rectangle $[0, 3n] \times [0, n] \cap \mathbb{Z}^2$. Prove that $\mathbb{P}_p(A_n) \to 1$ as $n \to \infty$.

Hint. First observe $\mathbb{P}([0,n]^2 \cap \mathbb{Z}^2$ intersects an infinite activated component$) \to 1$ as $n \to \infty$. Convince yourself that if $[0,n]^2 \cap \mathbb{Z}^2$ intersects an infinite activated component then (at least) one of the four overlapping rectangles forming the annulus $[0,3n]^2 \setminus ([(n+1),2n-1]^2) \cap \mathbb{Z}^2$ has an activated crossing in the shorter direction. Finally use Corollary 2.1 to conclude the proof.