Stochastic networks
Problem set 5
Due date: November 29, 2011

Exercise 1
Let $n \geq 1$ and let $(\Omega_n, \mathcal{A}_n, \mathbb{P}_n)$ be the probability space defined in the lecture. Furthermore, for $1 \leq k \leq n$, let $X_k : \Omega_n \to \{0, 1\}$ denote the projection on the $k$-th component. Which of the following events are increasing, decreasing, neither of the two?

(a) $|\{k : X_k = 1\}| = 5$
(b) $\forall 1 \leq k \leq n : (X_k = 1 \Rightarrow \forall j : (1 \leq j \leq k - 1 \Rightarrow X_j = 0))$
(c) $A_0$, where $A \in \mathcal{A}_n$ is decreasing

Hint. The answer may depend on the value of $n$.

Exercise 2
Let $n \geq 1$ and let $(\Omega_n, \mathcal{A}_n, \mathbb{P}_n)$ be the probability space defined in the lecture. Let $f, g : \Omega_n \to [0, \infty)$ be bounded increasing functions (i.e. $f(\omega) \leq f(\omega')$ for $\omega \leq \omega'$).

(a) show that $\mathbb{E}(fg) \geq \mathbb{E}(f)\mathbb{E}(g)$.
(b) use part (a) to give a more conceptual proof of exercise 2 in problem set 1.

Exercise 3
Consider Bernoulli bond percolation on $\mathbb{Z}^d$. Prove that the function $\theta(p)$ is continuous from the right.

Hint. The relation $\theta(p) = \lim_{n \to \infty} \mathbb{P}_p(o \to \partial B(n))$ may be useful, where $\partial B(n)$ is the set of vertices in $\mathbb{Z}^d$ that can be reached from the origin $o$ in at most $n$ but not in $n - 1$ steps. Furthermore you may use (without proof) that any decreasing limit of continuous functions is upper semicontinuous.

Exercise 4
Consider Bernoulli bond percolation in $\mathbb{Z}^2$ and let $p > p_c$. Denote by $A_n$ the event that there exists an activated vertical crossing of the rectangle $[0, 3n] \times [0, n] \cap \mathbb{Z}^2$. Prove that $\mathbb{P}_p(A_n) \to 1$ as $n \to \infty$. 
*Hint.* First observe $\mathbb{P}([0, n]^2 \cap \mathbb{Z}^2$ intersects an infinite activated component) $\to 1$ as $n \to \infty$. Convince yourself that if $[0, n]^2 \cap \mathbb{Z}^2$ intersects an infinite activated component then (at least) one of the four overlapping rectangles forming the annulus $[0, 3n]^2 \setminus \(((n + 1), 2n - 1]^2) \cap \mathbb{Z}^2$ has an activated crossing in the shorter direction. Finally use Corollary 2.1 to conclude the proof.